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FEBRUARY 1956

The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

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MATHEMATICS AS A PROFESSION TODAY IN INDUSTRY*

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INTRODUCTION

The word "Today" in the subject assigned to me implies that there has been a change in the status of mathematics in industry. This is true.

On the score of numbers alone, the change is impressive. At the beginning of this century the number of men in industry who could be described as mathematicians was negligible. Steinmetz was with General Electric—he had a Ph.D. in mathematics—and Campbell was with the Bell Telephone System. Perhaps there were others, though not many. Even in 1940, when I made a serious attempt to estimate the number, the figure I arrived at was only 150 in industry and government combined.[1] Last year, in a conference in New York on Training in Applied Mathematics, the present need, if not the actual number employed, was estimated to be around 1500.[2]

This rapid growth has brought new problems to the universities. As long as only an occasional mathematician entered industry, the curriculum could properly be planned without regard for him. Now that a significant percentage of graduates are industry bound, this is no longer true.

My object in this address will be to examine the place of the mathematician in industry, and bring out some of the basic requirements for his proper education.

We could make some progress toward these ends by taking a cross-section of the professional group as it exists today, noting the sources from which it was derived, the training of its members, and the variety of its activities; and drawing such conclusions as seem appropriate. This is what I attempted to do in 1940, with, I believe, useful results.[1] But in seeking light on educational needs it has a serious drawback. We do not educate *today's* mathematicians today, we educate *tomorrow's*, and our curriculum should be adapted to tomorrow's needs. On the present occasion, therefore, it will be more profitable to focus attention on the changing scene. We will thus understand better why the demand for industrial mathematicians is now so heavy whereas formerly it was not; why it developed when it did; and what the character of the future demand will be.

THE CHANGING SCENE

The young man who enters college next month at age 18 will, if present

* An address delivered at the request of the program committee at a meeting of the Mathematical Association of America, Ann Arbor, Michigan, August 29, 1955.

[1] "Industrial Mathematics," a report for the National Resources Planning Board. Published as part of "Research—A National Resource—II," a House Document, 77th Congress.

[2] "Applied Mathematics as a Responsibility of the Mathematical Profession." Published as part of "Proceedings of a Conference on Training in Applied Mathematics", sponsored by the American Mathematical Society and by the National Research Council under Contract NSF-C7 with the National Science Foundation.

industrial practices continue, retire 47 years from now at age 65. We would like to know what these 47 years will be like, and what demands they will place on his education. But man is strangely endowed; in three dimensions he can see what is ahead, but in the fourth, which is the dimension of time, only what is behind. So the best I can do is to look backward 47 years, and compare the world of 1908 with that in which we live today. By a process of extrapolation we can then form a judgment of what lies ahead.

Progress in science. In physics, in 1908, a lively subject of conversation was the special theory of relativity, which Einstein had formulated in 1905. Another was the quantum hypothesis, which Planck had published in 1901. But neither subject had yet had much impact on science. In fact, Einstein's general theory did not appear until 1914, and Bohr did not propose his model of the hydrogen atom until 1913. Millikan measured the charge on the electron in 1912; until then the existence of a unique and indivisible electric particle was in the realm of speculation.

These were the first stirrings of the revolution in physics which led to our present conceptions of atomic structure. Nuclear physics was to come even later. Isotope, energy level, nuclear spin, forbidden state—such words were not in the language when our student entered college in 1908. What physicist had yet heard of an eigenvalue? Or how many mathematicians either, for that matter?

In chemistry the change has been almost as marked. The combining weights of the elements had, of course, been measured, and when arranged in order of magnitude they looked too much like an arithmetical sequence to be ascribed to chance. But the deviations, too, were solidly established, and irritating because they were not understood. Moseley's paper on Atomic Numbers would not appear until 1914; and Aston's invention of the mass spectrograph, which gave the final confirmation, was five years later. The elements had also been arranged in the periodic table, in terms of their valences. The pattern of chemical behavior which this revealed was consistent in too many ways to be attributed to chance. But here also there was an irritating mystery, multiple valences. The Braggs had just seen those great spots-before-the-eyes of science, the scattering of X-rays by crystals. Eventually these would make the geometry of chemical bonds real and precise, so that the structural formulas of chemistry would no longer be limited to rows of letters. But in 1908 chemical valences and chemical bonds were still empirical facts with almost no systematic theory to tie them together. Except for a few dyestuffs, synthetic chemicals were unknown.

In communication, there were automobiles, telegraph and telephone; and the Wright brothers had made their historic flight in 1903. But flight was in its infancy, and no one had yet talked across the American continent, much less across the oceans. De Forest had just patented his vacuum tube in 1907, so of course there was no electronics.

In biology, Mendel's law was known. But there was as yet no talk of genes

or chromosomes; so of course the mechanism of inheritance and mutation was not understood. And the viruses and vitamins, which are newspaper words today, were then unknown.

Biochemistry was not even born. We can date its beginnings from the discovery of the protein building blocks by Fischer in 1913. Even the final identification of chlorophyl as the carrier between light and life dates only from 1913.

Who, in mathematics, had ever heard of Markoff processes, or the theory of games; of information theory or the theory of network synthesis?

The fact is, that in the last 47 years the whole fabric of physical science has changed. As one of the students who entered college in 1908, I have seen these changes. This is the exciting world, at that time still unknown, for which my college and university were asked to prepare me.

Progress in industry: New things. The technology which is founded on the physical sciences has shown equally radical progress. Its most obvious tokens are the new things we make: cellophane and nylon hose, TV sets and deep freeze units, aeroplanes, supersonic drills and atom bombs; transistors, electron microscopes and tracer isotopes. We cannot fail to marvel at the rate at which such new things have appeared, and the changes they have brought about in our standard of living.

These new things are ubiquitous, but they do not of themselves account for the growing demand for industrial mathematicians. In 1908 the makers of automobiles, locomotives and ocean liners felt no need to hire mathematicians; for the most part, in fact, they still do not. Why then are the makers of airplanes clamoring? Naval rifles and machine guns were designed by engineers; why is the mathematician needed to design a guided missile? Your home radio and TV sets are fine examples of what the engineer can do; why then was it necessary to develop a new branch of mathematics before a coaxial repeater could be built to amplify the signals of TV?

Better things. A second change in technology is the rising standard of performance which we demand of the products of industrial research. A few weeks ago I visited the oldest maker of submarine cables in the world, and watched the manufacture of the cable which will shortly carry 36 telephone conversations across the Atlantic. The precision required in this process was so great that a new factory, designed from the ground up to meet the stated demands, had been necessary. As one small example, instruments were provided to monitor continuously both the thickness and the dielectric constant of the plastic insulation as it came from the extruding press; and a servomechanism to adjust the speed of extrusion automatically so that the electrical capacity of the cable would be everywhere the same. Why was this precision necessary? Why must the dimensions of the copper wires and tapes be held to closer limits than ever before, and even the molecular weight of the plastic be controlled? If these things were not done, a submarine telephone cable might still be possible; but it would carry 30 or 25, or maybe even less messages; and the cost of the service rendered would be greater.

This example is unusual in degree perhaps, but not in kind. Servo control of machines is common in manufacturing today; so is the careful specification of primary materials. But in 1908 no one could have controlled the molecular weight of a polymer—in fact no one could even have measured it. Nor could the servo system have been designed—it was 25 years later before an adequate mathematical theory for that purpose was worked out.

It is natural that, as science moves forward and provides new knowledge and new techniques, industry finds profitable ways to use them. But only to the extent that the men of industry are alert to the new knowledge and competent to apply it. So the rising standard of performance puts requirements not only on materials and machines, but also on the men who design the products and processes. It requires them to bring to bear on each problem the entire wealth of modern science and technology, so far as it is pertinent, and to do this in a way which is imaginative without being impractical.

Frequently this is more than any man can do. This brings us to one of the most fundamental reasons why industry needs mathematicians, but we shall pass the point for the moment. It will arise again soon in an even more challenging form.

The system. The third important change in technology is the trend to develop systems to meet total situations, rather than separate functional components.

The cable, to which we referred a few moments ago, is actually part of such a system, the total purpose of which is to provide reliable transatlantic telephone service in adequate volume and at reasonable cost. The cable alone does not provide such service. There must be repeaters to amplify the signals at intervals and restore their proper wave form; terminal apparatus to generate the signals at the sending end and distribute them to the proper destinations at the receiving end; adequate and reliable power plants; equipment for testing, monitoring and adjusting the working apparatus, *etc.* All these functional units are interdependent; the shortcomings of one may to some extent be compensated by imposing stronger requirements on another. And all are influenced by the form in which the speech signals are encoded and many similar matters; for example, the use of pulse coded transmission would make it easier to restore the proper wave form of the signals, but would require a more extravagant frequency bandwidth; a vocoder system would conserve bandwidth at the expense of reduced clarity and more complicated terminal equipment.

The problem then, is to get the combination of parts which will give the best answer to the over-all problem. This may require a cable with requirements more stringent than the cable manufacturer likes, and less stringent than the repeater engineers would prefer. But if the system is wisely planned, no requirement will be so severe as to be technologically unreasonable; and those which are difficult to meet will assure advantages to the over-all performance of the system which are commensurate to their cost.

An even better example of system design is NIKE, the antiaircraft defense

system which is now being installed around all major cities. Here the problem was to protect strategic land locations against high-flying bombers carrying modern weapons of destruction. Before any part of this system was designed—even before design of any part was authorized—a careful study was carried out to formulate the problem clearly, and then determine whether a satisfactory solution was possible in the existing state of technology, and if so, by what means. Such a study had to take account of probable future developments in the enemy's aircraft and bombs; it had to look into the relative merits, for the case at hand, of homing and-guided missiles; what could be expected in propulsion, in air frame design, in radar accuracies, in jamming techniques, and a multitude of other matters. Only when this study was completed was the development of the NIKE system undertaken. As it happened, not a single functional part of the system was taken from what the market provided. Air frame, warhead, propulsion, radars, computers, even the vehicles in which the equipment is housed, were all newly designed to contribute their proper share to the over-all performance of the system.

The team. Let us look for a moment at the breadth and depth of technical competence which was necessary to carry out this study. Among the more obvious things are all the skills that go into the design of radars, computers, jet engines, air frames, and so on. Even at this point one can say that no one man can be expertly informed about all. There are also less obvious matters: the theory of games for example. It is clear that the enemy will not cooperate by flying a desired course or even passively follow his own original plan in the presence of unanticipated danger. What strategy should the NIKE missile then follow to win, taking full account of its own limitations on maneuverability, fuel capacity, *etc.*, as well as the probable limitations of the enemy?

I could enlarge this list further, but I think the point has been made. The development of a system—and even more importantly the planning of such a development—requires detailed knowledge of too many scientific fields to be the work of any one man. It requires a *team of experts*.

Research teams are not unusual today, either in the universities or in industry. In 1908 it was different. They were not necessary then; today they are. And the reasons why they are now necessary have, I hope, emerged from what I have said so far. Essentially there are three: the scope of science has greatly increased; we demand the higher standard of performance which modern science makes possible; and we are increasingly developing systems to meet a total situation, rather than machines to perform some basic function.

I can put this more briefly. We know more, therefore we can do more. But it is *collectively* that we know most, and hence *collectively* we must act to accomplish most.

Why mathematicians? Mathematicians are useful members of such teams. Why? The answer which trips off our tongue most glibly is, "Because there are problems to be solved."

This answer is wrong. Not merely inadequate, mind you. It completely

misses the essential point. Mathematicians do sometimes solve problems, in industry as elsewhere. But they are usually not good at it. Certainly not as good as machines. And certainly the solving of problems is not often of such central importance as to require special representation on a research team.

The indispensable function of the mathematician is formulating problems, not solving them. This is even true of the mathematician who specializes in computation, as one of the outstanding members of our mathematical group does. As scientists from all parts of our organization bring problems to him for calculation, the first discussion almost invariably concerns the origin of the problem and its formulation; and it is not unusual for the scientist to go away convinced that he had brought a different problem than he had intended.

It is because the mathematician is expert in analyzing relations, in distinguishing what is essential from what is superficial in the statement of these relations, and in formulating broad and meaningful problems, that he has come to be an important figure in industrial research teams. Obviously, these traits are most needed at the early stage where a situation is being studied and plans are being laid. He plays much less part in the execution of these plans. NIKE is a case at point: the study and planning of this project required a team of experts from many engineering areas—propulsion, aerodynamics, radar, digital circuitry, *etc.*—as well as from the military. But the central members of the team were the mathematicians. This was not merely because the theory of games was involved. It was principally because they, better than other scientists, are adept in detecting the essential thread that lies obscured by the irrelevant details and divergent languages of the other sciences. Later, as the development progressed, they played a less prominent part in its activities.

Not every development project separates itself as neatly into a study and an execution phase as did NIKE; and the role of the mathematician is not always so central. More often than not he is a consultant to other members of the team, rather than their leader. But the service which he performs is generally that of critical analyst and expert formulator of situations and concepts.

LOOKING AHEAD

We may now try to extrapolate into the future, and thus appraise the educational job that should be done today.

We do not know what the future will be in detail. We do know it will be different. Hence, our job is surely to educate men as best we can to meet that *different* future, whatever it may be.

We know that science will progress and probably with great rapidity. This is inevitable, because tools now available permit studies that in the past would have been impossible. To mention only a few, there are such familiar things as electron microscopes, tracer elements and supersonics, the possibilities of which have not yet been fully exploited. The study of materials at low temperatures, where the voice of the nucleus can be heard above the thermal noise, has only just begun. And, especially in chemistry and biochemistry, we have as yet

caught only a glimpse of the importance of pure chemicals.

We can be reasonably sure of some of the directions that scientific progress will take. The nature of chemical bonds will some day be understood completely enough and precisely enough to predict untried reactions with assurance. The relation between chemical structure and physical properties will also be understood, so that elasticity, tensile strength, electrical resistivity, and the like, can be computed for a substance which has not yet been synthesized. It may even happen, some day, that this process will be operated in reverse; the customer having specified the physical characteristics which he needs, a new molecule will be computed which will have them.

We can be reasonably sure that life processes will soon be better understood. We already know that the fundamental chemical changes in cells are profoundly influenced by minute quantities of chemicals in their environment. These chemical processes cannot be studied in the test tube because sufficiently pure chemicals are not available. In the last few years a few substances—germanium, silicon and indium—have been purified to the point where less than one foreign molecule in a billion remains; but these are not of great importance to biochemical research. When ways are found to purify those which are more significant, progress will be almost explosive.

Industry too will progress. The pushbutton factory is already here. I know a large one which operates round the clock; the total number of so-called productive employes is six—two for each shift! In the jargon of the day this is "automation." What its future will be no one can now foretell. But it has arrived in conjunction with two other events of major historical importance which make its rapid growth inevitable. One is the control of nuclear reactions, which can provide limitless quantities of easily transportable power. Like every major increase of available power in the past, it will both reduce the demand for muscular power and increase the products of labor. The other is the creation of solid state devices such as ferrites, ferroelectrics, and transistors. The reliability and low wastage of power inherent in such devices will make it practicable to construct communication systems of capacity hitherto unknown, and control devices of almost fantastic complexity. One consequence will surely be decentralization in ways now scarcely dreamed of. For example, we may expect the parts of a single push-button factory to be widely dispersed, and yet operate automatically under a central system of control.

But all this means that the industrial problem of developing a product, and designing a factory to produce it, will become even more complex. It will clearly be, even more often than today, a problem of integral system design. Only thus can full advantage be taken of the possibilities opened up by the advances in science. And only those who are masters of science in all its avenues can fully exploit its possibilities. So quite clearly the job will be even more beyond the capacities of any one man than now, and the team will be an even greater necessity.

Again, we may put this briefly: In the future we shall know more, therefore

we shall be able to do more. But since we shall only know *collectively*, we shall have to work *collectively*.

What sort of mathematician? Let us attempt to visualize this team of the future, which we must educate today, and especially the mathematician on it.

There is certainly nothing to suggest that he will be less wanted than today. It is not an accident that the great increase in industrial demand for mathematicians in the last 15 years coincided with the rapid development of team research. It will continue to grow because the industrial exploitation of advancing science will demand ever more competent teams, and the teams will not be fully effective without the mathematician. There may even be relatively more situations like NIKE, in which he will play a central role; though I believe his position will be advisory more often than controlling.

The team will need to understand more things, and its individual members will need to understand them more thoroughly. To put it more briefly, they will have to be better educated. If anyone is inclined to feel that this is a speculative judgment, I need only point out that industry is hiring today a larger proportion of Ph.D.'s than ever before. Even on its engineering staffs, such men are needed in numbers that are not available. I know of companies, both here and abroad, which are hiring hundreds of engineering graduates and giving graduate education while on the company payroll.

Better educated teams, however, mean faster company for the mathematicians on them. If there is any doubt that industrial mathematicians are playing in the big league today, there can be none about tomorrow. So in addition to the increased breadth that will be necessitated by the rapid development of science, they will need increased depth and versatility to keep pace with their associates.

This is a challenging situation. Our present patterns of education for Ph.D.'s do not produce industrial mathematicians of the breadth of scientific outlook needed today. They are even less adequate for the tomorrow in which our graduates will live. Clearly the Ph.D.—today's Ph.D.—is not enough.

It is not a unique problem. The same outlook faces the other physical sciences as well. But except for engineering—which has failed even to provide Ph.D. training for more than a negligible proportion of its students—the other sciences are alert to the need. In mathematics there is still a widespread misunderstanding that the industrial mathematician is just a problem solver.

Let us not make that mistake. We are talking about problem formulators, very superior consultants who can play in big league company. That is what industry needs. It cannot, without serious penalty, settle for less.

What sort of man should he be? He must be experienced and adept in the basic thought processes of mathematics, and of course he must also have a working knowledge of its various branches. These are the usual requirements of a mathematician anywhere.

Beyond this is the requirement that he be able to work effectively as a member of a team. To do so, he must have broad scientific interests and enough

knowledge of other sciences to enable him to deal intelligently with his associates; and he must have a personality suitable for joint effort.

I expanded on these ideas in the previous paper to which I have referred,[1] and need not go into them more particularly today. There is, however, one statement from that paper which I would like to quote verbatim:

> "He must be a man of outstanding ability. No one wants the advice of mediocrity. Among industrial mathematicians there is no place for the average man."*

CONCLUSION: THE EDUCATOR'S JOB

The educator's job today is nowise different from that of 1908. It is to fit the student for the life he will lead; life in a future as yet unknown, and probably as different from the present as was the case in 1908.

If we are to train men to meet this new and challenging future, we must train them in those things which are most lasting in our science—its habits of thought, its fundamental disciplines, its peculiar relation to philosophy and to the other sciences. Such training would not be meaningful if it did not deal with specific theorems and specific applications; these can only be chosen from the problems of today and yesterday; but their purpose should not be mistaken; it is to illustrate certain habits of thought and to train in certain disciplines; it is to illuminate the problems of tomorrow, not to solve the problems of today.

As I see it, just two things are essential to guarantee this result. The first is a clear appreciation of the need. Not merely that "mathematicians" are wanted, but what sort of service they will perform. Given this understanding of the need, I will trust our educators to build the curriculum to meet it.

The second is time. Even today our young men are likely to be in their late twenties by the time they get their already inadequate Ph.D. If we seek to make this adequate by adding more years to the curriculum, we will consume even more of their imaginative youth. We should seriously ask if there is not a better way.

I think there is. It is the atavistic way. We must compress what is essential from the past of our science into fewer years, so that its present can be reached at an acceptable age. This will not be easy. Not that it is inherently difficult, but it runs counter to other educational purposes which have been more widely

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* Lest this be understood to mean that the world has no important place for the man of average, or less than average, training or ability, I may also quote from my address before the Conference on Training in Applied Mathematics:[2]

> "It is important to recognize that . . . industrial and government employers . . . need men with more than one level of mathematical training—from technicians who are skilled in performing certain mathematical operations, to creative scientists who can invent and improve their art."

I have no doubt that industry will need many computers, and will want them even better trained. They also will deserve the respect and dignity due all useful members of society. But my present address is concerned with scientists; not with technicians.

publicized and are generally—I think too generally—accepted. What I have in mind is a very radical reappraisal of the educational values of mathematical subjects, and their psychological adaptability to various ages. Why, for example, do we not teach algebra along with, or before arithmetic? Is it harder to understand that $a+a=2a$ than that $1+1=2$? At how early an age can imaginary numbers be introduced? Does not a child acquire the ability to appreciate (or is it accept?) such simple logical relationships at an earlier age than he acquires the experience and judgment needed for many problems of arithmetic? If these things were done, could we not teach calculus to ninth year students? And could we not teach it in the complex plane at the start? Finally, could we not do all these things without devoting more hours to mathematics than is now devoted by those who take it at all as a high school elective?

I can see no less radical way to get the time that is needed to educate mathematicians for industry, and still preserve their most valuable creative years for productive work. It will not be easy to bring about. But the stakes are worth it; for such an atavistic approach to mathematics is as badly needed by students who enter engineering or the physical sciences as by the mathematicians themselves.

A MEDIEVAL ITERATIVE ALGORISM

E. S. KENNEDY and W. R. TRANSUE

Brown University, The Institute for Advanced Study, The American University of Beirut, and Kenyon College

A standard interpolation expedient of ancient astronomy consisted of putting $f(\theta)=k \sin \theta$ to obtain general values of a periodic, symmetrical function of which it was known only that $f(0°)=f(180°)=0$, and $f(90°)=k$. This device is at least as old as Ptolemy (*c.* 150 A.D.). The last column of his table for the computation of planetary latitudes* is in effect a table of cosines, to be used as indicated above. The table is a true interpolation scheme in the sense that it was set up without direct reference either to the underlying astronomical phenomenon or to the complicated mathematical idealization to which it yields an approximation.

Of course Ptolemy had no knowledge of the sine or cosine functions as such. His interpolation function was obtained from a table of lunar latitudes,† itself computed by use of a table of chords.‡

* The Almagest, Book 13 (Syntaxis mathematica, ed. J. L. Heiberg, 2 vols., Leipzig, 1898–1903; German translation by K. Manitius, 2 vols., Leipzig, 1912–13). The edition and French translation hereafter referred to is by Halma, Composition mathématique . . . , Paris, 1813–1816.

† *Ibid.*, Halma, vol. i, p. 316.

‡ *Ibid.*, Halma, vol. i, pp. 38–45.

The Hindus, who invented the ancestor of the modern sine function, made extensive practical application of this style of interpolation. The Hindu sine (written here Sin with a capital S to distinguish it from the modern function) is definable as $\text{Sin}\,\theta \equiv R \sin\theta$, where R is the radius of the defining circle.

Various values for R were current. Where numbers were customarily represented in a place-value system with base sixty, it was convenient, following the lead of Ptolemy, to put $R=60$. With this arrangement, standard with the Arabic-writing Islamic astronomers, Sin θ has the same sexagesimal digits as $\sin\theta$, but having the "sexagesimal point" moved one place over.

An R ascribed§ to the Hindu Āryabhaṭa is 3438, obtained by putting the linear unit equal to one minute of arc length along the circumference of the defining circle. If now θ is in minutes, $\text{Sin}\,\theta \approx \theta$ for small θ, a property resembling the corresponding one for radians.

A third value of R is 150, probably so set because with this parameter Sin ϵ very nearly equals the sexagesimal base, ϵ being the inclination of the ecliptic.‖ If θ is the longitude of a point on the ecliptic, its declination δ is given by the expression $\sin\delta = \sin\epsilon \sin\theta = k \sin\theta$. Thus a table of declinations can be used as the basis for an interpolation scheme which resembles, but is not identical with that described in the first paragraph.¶

For the representation of smooth periodic phenomena with the maximum not midway between the zeros, say at $\theta = 90° - m$ for $0° < m < 180°/\pi$, a natural modification of the standard procedure would be to effect some continuous transformation of the θ's leaving the endpoints ($\theta = 0°$ and $\theta = 180°$) undisturbed and displacing the midpoint ($\theta = 90°$) an amount m to the left (see the figure). Such a transformation is defined by the expression

$$t = \theta - m \sin\theta, \tag{1}$$

and the function

$$\Phi(t) = k \sin\theta(t) \tag{2}$$

has the desired property. Tables of this $\Phi(t)$ exist in two Islamic astronomical handbooks** in connection with the determination of lunar parallax components. Neither author motivates the method, and one of them clearly indicates that it

§ *Cf.* Rasā'il ul-Bīrūnī (in Arabic), Osmania Oriental Publications Bureau, Hyderabad-Dn., 1948, p. 178; also Burgess, E., Translation of the Sūrya-Siddhānta, a Textbook of Hindu Astronomy, . . . , reprinted from the edition of 1860, Univ. of Calcutta, 1935, pp. 59–64.

‖ Tables containing this function appeared in medieval Europe; *cf.* for example, Millás-Vallicrosa, Estudios sobre Azarquiel, Madrid, 1943–1950, p. 44; and Neugebauer and Schmidt, Hindu astronomy at Newminster in 1428, Annals of Science, vol. 8, 1952, pp. 221–228.

¶ The tables of solar and lunar equations in Die astronomischen Tafeln des . . . al-Khwārizmī (edited by Björnbo, Besthorn, and Suter, Copenhagen, 1914, pp. 132–136) were computed by this method.

** *Ibid.*, pp. 191–192, and Bodleian (Oxford) Ms. Seld. A. 30, Az-Zīj al-Jadīd li Ibn ash-Shāṭir. For the astronomical setting of the problem, see Kennedy, E. S., "Parallax theory in Islamic Astronomy," to appear in *Isis*.

is not his invention. From the fact that most of the material in the other is known to be of Hindu origin it is conjectured that this technique also was worked out in India.

A third Arabic work, written in the ninth century by *Ḥabash al-Ḥāsib al-Marwazī*†† (*i.e.*, The Computer from Merv, in Turkestan) gives a neat recursion relation for determining Φ. He prescribes the formation of the sequence

$$\begin{aligned} &\theta_0(t) = t + m \sin t, \\ &\cdots\cdots\cdots\cdots \\ &\theta_n(t) = t + m \sin \theta_{n-1}(t), \\ &\cdots\cdots\cdots\cdots \end{aligned} \qquad (n = 1, 2, 3, \cdots) \tag{3}$$

the limit of which satisfies (1). Hence

$$\Phi(t) = k \sin \lim_{n\to\infty} \theta_n(t). \tag{4}$$

The symbolism is, of course, modern; in all of the sources the processes are written out as verbal statements.

Ḥabash gives no proof, contenting himself with the statement that the desired function is

$$k \sin \theta_3(t).$$

In fact the sequence does converge, and with sufficient rapidity that θ_3 yields a fair approximation to the tables.

We show briefly the convergence of (3), or more generally of the sequence

$$\begin{aligned} &\theta_0(t) \equiv 0, \\ &\cdots\cdots\cdots\cdots \\ &\theta_n(t) = t + mS[\theta_{n-1}(t)], \\ &\cdots\cdots\cdots\cdots \end{aligned} \tag{5}$$

where the function S satisfies a Lipschitz condition

$$|S(t_1) - S(t_2)| \leqq k|t_1 - t_2| \quad \text{with} \quad k < 1/m \quad \text{and} \quad S(0) = 0.‡‡$$

Since

$$|\theta_n(t) - \theta_{n-1}(t)| = m|S[\theta_{n-1}(t)] - S[\theta_{n-2}(t)]| \leqq mk|\theta_{n-1}(t) - \theta_{n-2}(t)|,$$

we have

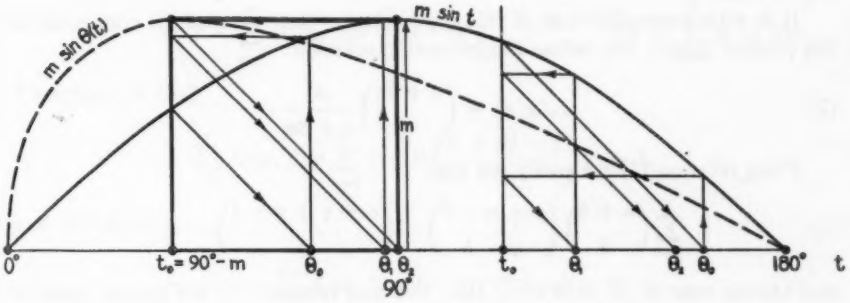
$$\left|\frac{\theta_n(t) - \theta_{n-1}(t)}{\theta_{n-1}(t) - \theta_{n-2}(t)}\right| \leqq mk < 1,$$

†† Manuscript 784,2° in the Yeni Jami library in Istanbul. Ḥabash was one of the scientists attracted to the Baghdad and Damascus courts of the Caliph al-Ma'mūn.

‡‡ This follows directly from, for example, Kantorovich, Vulikh and Pinsker, Functional Analysis in Partially Ordered Spaces, Moscow, 1950 (in Russian), p. 467, Cor. 2.24.

so that the ratio test shows the convergence of the series with general term $\theta_n(t)-\theta_{n-1}(t)$, and hence of the sequence $\theta_n(t)$.

From the point of view of the historian it is of interest to ask not only about the validity of the process, but also who invented it and how. Neither question can at present be satisfactorily answered. There is no particular reason for thinking that Ḥabash was the inventor. It is natural to suspect a Hindu origin for the algorism, by extrapolation from the fact that the techniques in which this section of Ḥabash's work is embedded are demonstrably Hindu. The nature of the operation provides no clue, for computations by successive approximation were used by Hellenistic, Hindu, and Muslem scientists alike. As for a reconstruction of the process of invention, it will be illuminating to consider a graphical method of generating the sequence $\{\theta_n\}$. In the figure below, and for any



fixed t_0, draw a line with slope -1 from point $(t_0, m \sin t_0)$ to intersect the horizontal axis at the point marked θ_0. From θ_0 erect a perpendicular meeting the curve in a point, and project the latter horizontally to P on the ordinate through t_0. In turn project P downward to the right as shown to obtain θ_1, and so continue the cycle of projections. It can be shown that for $t_0=90°-m$ the bounded sequence $\{\theta_n\}$ is monotonic increasing, converging to 90°. Hence $\sin\theta(t)$ has its maximum at $t=90°-m$. It is submitted that the unknown inventor, thoroughly familiar with the technique of trigonometric interpolation, but desiring a function of the nature of the curve shown dotted in the figure, set up the equivalent of expressions (1) and (2). This gives the desired type of function, but has the disadvantage that, given a t, one cannot directly obtain a corresponding Φ. A direct method being desirable, the formal similarity between expression (1) and

$$t = \theta_n(t) - m \sin \theta_{n-1}(t)$$

may well have been noticed, and the convergence of the implicit sequence concluded from considerations analogous to our graphical method.

SOME GENERALIZATIONS OF VANDERMONDE'S CONVOLUTION

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Among the elegant results implied by the binomial theorem, one of the most attractive and widely known is Vandermonde's convolution, which states that:

$$\sum_{i=0}^{k}\binom{r}{i}\binom{q}{k-i}=\binom{r+q}{k}. \tag{1}$$

This formula is valid for all integral values of r and q. Since each member is a polynomial in r and also in q of degree k, it follows that (1) is a polynomial identity in r and q.

It is with generalizations of this type relation that we shall be concerned in the present paper. We define the general expression:

$$A_n(\alpha,\beta)=\binom{\alpha+\beta n}{n}\frac{\alpha}{\alpha+\beta n}. \tag{2}$$

From relation (1) we easily see that

$$\sum_{k=0}^{n}\binom{\alpha+k}{k}\binom{\gamma+n-k}{n-k}=\binom{\alpha+\gamma+n+1}{n}, \tag{3}$$

and taking note of (2) it is clear that this and relation (1) are special cases of the following when $\beta=0$ and when $\beta=1$.

$$\sum_{k=0}^{n}A_k(\alpha,\beta)A_{n-k}(\gamma,\beta)=A_n(\alpha+\gamma,\beta). \tag{4}$$

In order to demonstrate (4) we may proceed as follows, developing certain preliminary results first.

Taking the nth difference of a polynomial of degree less than or equal to $n-1$ we must have zero for $n\geqq 1$. Hence we have

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{\alpha+k\beta}{n}\frac{\alpha}{\alpha+k\beta} \tag{5}$$

$$=(-1)^nA_n(\alpha,\beta)+\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}\binom{\alpha+k\beta}{n}\frac{\alpha}{\alpha+k\beta}=0,\ n\geqq 1.$$

Hence

$$A_k(\alpha,\beta)=\sum_{j=0}^{k-1}(-1)^{j+k+1}\binom{k}{j}\binom{\alpha+\beta j}{k}\frac{\alpha}{\alpha+\beta j},\qquad k\geqq 1. \tag{6}$$

This relation enables us to evaluate a certain infinite series as follows:

$$\sum_{k=0}^{\infty} A_k(\alpha, \beta)z^k = 1 + \sum_{k=1}^{\infty} A_k(\alpha, \beta)z^k$$

$$= 1 + \sum_{k=1}^{\infty} z^k \sum_{j=0}^{k-1} (-1)^{j+k+1}\binom{k}{j}\binom{\alpha+\beta j}{k}\frac{\alpha}{\alpha+\beta j}$$

$$= 1 + \sum_{j=0}^{\infty} (-1)^j \frac{\alpha}{\alpha+\beta j} \sum_{k=j}^{\infty} (-1)^k \binom{k+1}{j}\binom{\alpha+\beta j}{k+1} z^{k+1}$$

$$= 1 + \sum_{j=0}^{\infty} (-1)^j \binom{\alpha+\beta j}{j} \frac{\alpha}{\alpha+\beta j} \sum_{k=j}^{\infty} (-1)^k \binom{\alpha+\beta j-j}{k+1-j} z^{k+1}$$

$$= 1 - \sum_{j=0}^{\infty} \binom{\alpha+\beta j}{j} \frac{\alpha}{\alpha+\beta j} z^j \sum_{k=1}^{\infty} (-1)^k \binom{\alpha+\beta j-j}{k} z^k$$

$$= 1 - \sum_{j=0}^{\infty} A_j(\alpha, \beta)z^j \sum_{k=0}^{\infty} (-1)^k \binom{\alpha+\beta j-j}{k} z^k + \sum_{j=0}^{\infty} A_j(\alpha, \beta)z^j.$$

Therefore we find

$$\sum_{j=0}^{\infty} A_j(\alpha, \beta)z^j \sum_{k=0}^{\infty} (-1)^k \binom{\alpha+\beta j-j}{k} z^k = 1,$$

and this becomes

$$\sum_{j=0}^{\infty} A_j(\alpha, \beta)z^j(1-z)^{\alpha+\beta j-j} = 1.$$

Making changes in the notation and examining the convergence we find

$$\sum_{j=0}^{\infty} A_j(\alpha, \beta)z^j = x^\alpha, \tag{7}$$

where:

$$z = \frac{x-1}{x^\beta} \quad \text{and} \quad |z| < \left|\frac{(\beta-1)^{\beta-1}}{\beta^\beta}\right|.$$

Again, we also find the following:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{\alpha+\beta k}{n} = (-1)^n\beta^n, \qquad n \geqq 0, \tag{8}$$

which follows easily by finite difference arguments, and is well-known. From (8) we find:

$$(-1)^n\binom{\alpha+\beta n}{n} = (-1)^n\beta^n - \sum_{k=0}^{n-1} (-1)^k \binom{n}{k}\binom{\alpha+\beta k}{n}, \qquad n \geqq 1.$$

Hence

$$\begin{aligned}
\sum_{n=0}^{\infty}(-1)^n\binom{\alpha+\beta n}{n}z^n &= 1+\sum_{n=1}^{\infty}(-1)^n\binom{\alpha+\beta n}{n}z^n \\
&= 1+\sum_{n=1}^{\infty}(-\beta z)^n-\sum_{n=1}^{\infty}z^n\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}\binom{\alpha+\beta k}{n} \\
&= \frac{1}{1+\beta z}-\sum_{k=0}^{\infty}(-1)^k\sum_{n=k}^{\infty}\binom{n+1}{k}\binom{\alpha+\beta k}{n+1}z^{n+1} \\
&= \frac{1}{1+\beta z}-\sum_{k=0}^{\infty}(-1)^k\binom{\alpha+\beta k}{k}\sum_{n=k}^{\infty}\binom{\alpha+\beta k-k}{n+1-k}z^{n+1} \\
&= \frac{1}{1+\beta z}-\sum_{k=0}^{\infty}(-1)^k\binom{\alpha+\beta k}{k} \\
&\quad \left\{\sum_{n=0}^{\infty}\binom{\alpha+\beta k-k}{n}z^{n+k}-z^k\right\} \\
&= \frac{1}{1+\beta z}-\sum_{k=0}^{\infty}(-1)^k\binom{\alpha+\beta k}{k}z^k(z+1)^{\alpha+\beta k-k} \\
&\quad +\sum_{k=0}^{\infty}(-1)^k\binom{\alpha+\beta k}{k}z^k.
\end{aligned}$$

Therefore we have

$$\sum_{k=0}^{\infty}(-1)^k\binom{\alpha+\beta k}{k}z^k(1+z)^{\alpha+\beta k-k}=\frac{1}{1+\beta z}.$$

This may now be rewritten in the form

$$\sum_{k=0}^{\infty}\binom{\alpha+\beta k}{k}z^k=\frac{x^{\alpha+1}}{(1-\beta)x+\beta}, \tag{9}$$

where

$$z=\frac{x-1}{x^{\beta}}, \qquad |z|<\left|\frac{(\beta-1)^{\beta-1}}{\beta^{\beta}}\right|.$$

A discussion of the convergence of this last series may be found in Pólya and Szegö, *Aufgaben und Lehrsätze*, Springer Verlag, Berlin, 1925, first volume, problems 206 to 218; in particular problem 216, *zweiter Abschnitt*. These series may also be found as Pólya and Szegö indicate from the application of the Lagrange theorem, a statement of which is given in Whittaker and Watson, *A Course of Modern Analysis*, 4th Edition, 1927, page 133.

Now from relation (7) we have the following:

$$x^{\alpha}\cdot x^{\gamma} = x^{\alpha+\gamma} = \sum_{k=0}^{\infty} A_k(\alpha+\gamma,\beta)z^k$$

$$= \left(\sum_{i=0}^{\infty} A_i(\alpha,\beta)z^i\right)\left(\sum_{j=0}^{\infty} A_j(\gamma,\beta)z^j\right)$$

$$= \sum_{k=0}^{\infty} z^k \sum_{i=0}^{k} A_i(\alpha,\beta)A_{k-i}(\gamma,\beta).$$

Therefore we may compare coefficients and we find

$$\sum_{k=0}^{n} A_k(\alpha,\beta)A_{n-k}(\gamma,\beta) = A_n(\alpha+\gamma,\beta), \tag{10}$$

which is exactly relation (4), and since it is a polynomial expression on each side here of degree n in α, β, γ, it is an identity in them.

Just as we have derived (10) we may combine relations (7) and (9), and we find the following:

$$\frac{x^{\alpha+\gamma+1}}{(1-\beta)x+\beta} = \sum_{k=0}^{\infty}\binom{\alpha+\gamma+\beta k}{k}z^k = x^{\gamma}\cdot\frac{x^{\alpha+1}}{(1-\beta)x+\beta}$$

$$= \sum_{i=0}^{\infty}\binom{\gamma+\beta i}{i}\left(\frac{\gamma}{\gamma+\beta i}z^i\right)\sum_{j=0}^{\infty}\binom{\alpha+\beta j}{j}z^j$$

$$= \sum_{k=0}^{\infty} z^k \sum_{i=0}^{k}\binom{\gamma+\beta i}{i}\frac{\gamma}{\gamma+\beta i}\binom{\alpha+\beta(k-i)}{k-i}.$$

Comparing the coefficient of z^k in these expressions we find therefore that with appropriate changes in symbols,

$$\sum_{k=0}^{n}\binom{\gamma+\beta k}{k}\binom{\alpha+(n-k)\beta}{n-k}\frac{\gamma}{\gamma+\beta k} = \binom{\alpha+\gamma+\beta n}{n}. \tag{11}$$

This result is quite as valuable as relation (10) although the latter is more symmetrical. One more relation will be developed before some applications are made. Let

$$f(\alpha,x) = \frac{x^{\alpha+1}}{(1-\beta)x+\beta}.$$

Then,

$$f(\alpha,x)\cdot f(\gamma,x) = f(\alpha+\epsilon,x)\cdot f(\gamma-\epsilon,x),$$

so that by utilizing relation (9) we find at once

$$(12)\qquad \sum_{k=0}^{n}\binom{\alpha+k\beta}{k}\binom{\gamma+(n-k)\beta}{n-k}=\sum_{k=0}^{n}\binom{\alpha+\epsilon+k\beta}{k}\binom{\gamma-\epsilon+(n-k)\beta}{n-k},$$

which is again a symmetrical expression but not too useful.

Some Applications and Examples in the Literature

I. One of the unsolved problems in this MONTHLY revolves around this type of expansion being considered here. K. L. Chung proposed in problem no. 4211, (1946, page 397) the summation

$$(13)\qquad \binom{nd}{n}=d(d-1)\sum_{k=1}^{n}\frac{(dk-2)!}{(k-1)!(dk-k)!}\binom{nd-kd}{n-k},\qquad n\geqq 1,$$

which may be rewritten in the form

$$\binom{nd-1}{n-1}=\sum_{k=0}^{n-1}\binom{d-1+dk}{k}\frac{d-1}{d-1+dk}\binom{(n-1)d-kd}{n-1-k}$$

because (13) may first of all be rewritten as

$$\begin{aligned}\binom{nd}{n}\frac{1}{d-1}&=\sum_{k=1}^{n}\binom{dk}{k}\binom{nd-kd}{n-k}\frac{1}{dk-1}\\&=\sum_{k=1}^{n}\binom{dk-1}{k-1}\frac{d}{dk-1}\binom{nd-kd}{n-k}\\&=\sum_{k=0}^{n-1}\binom{dk+d-1}{k}\frac{d}{dk+d-1}\binom{nd-kd-d}{n-1-k}.\end{aligned}$$

But this is evaluated from relation (11) immediately if we first allow n there to be replaced by $n-1$, and let $d=\beta$, $\gamma=d-1$, and $\alpha=0$.

II. Another instance of this summation is in Torelli's sum. Torelli's result is quoted in the *Jahrbuch über die Fortschritte der Mathematik*, Volume 26, Jahrgang 1895, page 281. The original paper was entitled *Qualche formola relativa all' interpretazione fattoriale delle potenze* (Batt. G. XXXIII, pp. 179–182). We have here

$$x^{\overline{n}}=\prod_{j=0}^{n-1}(x+j)=n!\binom{x+n-1}{n}$$

$$(14)\qquad (x+y)^{\overline{n}}=\sum_{r=0}^{n}\binom{n}{r}y(y-rh+1)^{\overline{r-1}}(x+rh)^{\overline{n-r}}.$$

Or, therefore,

$$\binom{x+y+n-1}{n}$$

$$= (-1)^n \sum_{r=0}^{n} (-1)^r \binom{y-rh+r}{r}\binom{-x-rh}{n-r}\frac{y}{y-rh+r}$$

$$= \sum_{r=0}^{n} \binom{y-rh+r}{r}\binom{x-1+rh+n-r}{n-r}\frac{y}{y-rh+r}$$

$$= \sum_{k=0}^{n} \binom{y+(1-h)k}{k}\binom{x-1+nh+(n-k)(1-h)}{n-k}\frac{y}{y+(1-h)k},$$

which clearly follows from relation (11) when we put

$$\gamma = y, \qquad \alpha = x-1+nh, \qquad \beta = 1-h.$$

III. One of the often quoted relations of a convolution type is formula number 17, pages 64–68, of the *Synopsis der höheren Mathematik*, Volume I, by Johann Georg Hagen, Berlin, 1891. This was in German, although Hagen was a professor at Georgetown University for some time. Netto, in his *Lehrbuch der Combinatorik*, Teubner, Leipzig, 1901, devotes Chapter 13 to Hagen's combinatorial identities, and points out a few errors in some of the relations. In each case, reference is made by Hagen and Netto to some work in this field by Heinrich August Rothe, whose paper *Formulae de serierum reversione demonstratio universalis signis localibus combinatorio-analyticorum vicariis exhibita*, Leipzig, 1793, would seem to be one of the earliest references to these general convolutions.

In any event, Hagen's formula 17 may be derived in the following manner. From relation (7) we have

$$\sum_{k=0}^{\infty} A_k(\alpha, \beta) w^{p+qk} = w^p x^\alpha,$$

where

$$w = \frac{(x-1)^{1/q}}{x^{\beta/q}}.$$

On the interval of absolute convergence we may differentiate and we find

$$\sum_{k=0}^{\infty} A_k(\alpha, \beta)(p+qk) w^{p+qk-1}$$

$$= pw^{p-1}x^\alpha + \frac{\alpha w^p x^{\alpha-1} q x^{(\beta/q)+1}}{(x-1)^{(1/q)-1}\{(1-\beta)x+\beta\}}$$

$$= pw^{p-1}x^\alpha + \alpha q w^{p-1}\frac{x^{\alpha+1}}{(1-\beta)x+\beta} - \alpha q w^{p-1}\frac{x^\alpha}{(1-\beta)x+\beta}.$$

Therefore we have

$$(15)\quad \sum_{k=0}^{\infty} A_k(\alpha,\beta)(p+qk)w^{p+qk} = pw^p x^\alpha + \alpha q w^p \frac{x^{\alpha+1}}{(1-\beta)x+\beta} - \alpha q w^p \frac{x^\alpha}{(1-\beta)x+\beta}.$$

Now, we let

$$S = \left\{\sum_{k=0}^{\infty} A_k(\alpha,\beta)(p+qk)w^{p+qk}\right\} \cdot \left\{\sum_{j=0}^{\infty} A_j(\gamma,\beta)w^{p+qk}\right\}$$

$$= w^p \sum_{k=0}^{\infty} w^{p+qk} \sum_{i=0}^{k} A_i(\alpha,\beta)A_{k-i}(\gamma,\beta)(p+qi).$$

But, through the use of (15), this is equal to

$$S = w^p x^\gamma \left\{pw^p x^\alpha + \alpha q w^p \frac{x^{\alpha+1}}{(1-\beta)x+\beta} - \alpha q w^p \frac{x^\alpha}{(1-\beta)x+\beta}\right\}$$

$$= w^p \left\{p\sum_{k=0}^{\infty} A_k(\alpha+\gamma,\beta)w^{p+qk} + \alpha q \sum_{k=0}^{\infty}\binom{\alpha+\gamma+\beta k}{k} w^{p+qk} - \alpha q \sum_{k=0}^{\infty}\binom{\alpha+\gamma-1+\beta k}{k} w^{p+qk}\right\}$$

$$= w^p \sum_{k=0}^{\infty} w^{p+qk}\left\{\frac{p(\alpha+\gamma)}{\alpha+\gamma+\beta k} + \alpha q - \alpha q \frac{\alpha+\gamma+\beta k-k}{\alpha+\gamma+\beta k}\right\}\cdot\binom{\alpha+\gamma+\beta k}{k}$$

$$= w^p \sum_{k=0}^{\infty} w^{p+qk}\binom{\alpha+\gamma+\beta k}{k}\frac{p(\alpha+\gamma)+k\alpha q}{\alpha+\gamma+\beta k}.$$

But now we equate coefficients of these two expansions of S and the result is the relation

$$(16)\quad \sum_{k=0}^{n} A_k(\alpha,\beta)A_{n-k}(\gamma,\beta)(p+qk) = \frac{p(\alpha+\gamma)+n\alpha q}{\alpha+\gamma+\beta n}\binom{\alpha+\gamma+\beta n}{n}, \quad n \geqq 0,$$

which is a polynomial identity in α, γ and β. Letting $\gamma+\beta n=y$ and $\alpha=x$, we have

$$(17)\quad \sum_{k=0}^{n}\binom{x+\beta k}{k}\binom{y-\beta k}{n-k}\frac{p+qk}{(x+\beta k)(y-\beta k)} = \frac{p(x+y-\beta n)+nxq}{(x+y)x(y-\beta n)}\binom{x+y}{n},$$

which is Hagen's relation 17 (and number 17 here also). Clearly, relations (10) and (11) are the special cases already treated.

It would be interesting to construct a proof of these convolutions using finite series, the method of finite differences, or otherwise, just as Vandermonde's convolution relation (2) was obtained. We remark that Vandermonde's convolution is the special case when we let $\beta=0$. The equivalent form of Vandermonde's result is that obtained when $\beta=1$. It would be rather interesting to try to prove relation (10) for example by induction on β or on n. However, the resulting expressions do not seem to reduce very manageably.

DISTANCE SUMS ON A SPHERE AND ANGLE SUMS IN A SIMPLEX

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1. Introduction. It was shown in [1] that if S is the sum of the dihedral angles in a tetrahedron and T the sum of the trihedral angles then,

(A) $$T = 2S - 4\pi$$

and

(B) $$2\pi \leqq S \leqq 3\pi.$$

Equation (1) was obtained from the well known formula relating the area A of a spherical triangle and the sum $\sum$ of its angles,

(C) $$A = \sum - \pi.$$

This paper is concerned with generalizing relations (B) and (C). The methods of [1] do not appear to permit a complete generalization of (B) although they give the best upper bound for S and a non-trivial lower bound. This paper gives best upper and lower bounds for S. Furthermore, (C) does not generalize to an equation, but to an inequality.

In [1] I remarked that one would expect those results to be known. It seems appropriate to state here that they were. (See [2].) It should also be mentioned that results related to those of this paper have been investigated by T. S. Motzkin in [3], using different methods.

2. Terminology. By the *dihedral* angle between two $(n-1)$-dimensional hyperplanes in E_n is meant the supplement of the angle between their directed normals. Given n directed planes in E_n whose intersection is a point p, by the *multihedral* angle they determine we mean the surface content they cut out on the unit sphere S_{n-1} whose center is p. We suppose throughout that $n>2$.

Given the unit sphere S_{n-1}, with center p, every $(n-1)$-dimensional hyperplane through p determines a unit hypersphere S_{n-2} on the S_{n-1}. By the angle between two hyperspheres on S_{n-1} we mean the dihedral angle between their determining hyperplanes.

We are now in a position to formulate the problem precisely. A simplex in E_n consists, for our purposes, of $n+1$ linearly independent points and the $n+1$ hyperplanes they determine. We consider each hyperplane directed toward the point not lying on it. Then to measure a multihedral angle at one vertex, we have simply to measure the surface content of a spherical simplex on the unit S_{n-1} with the vertex as center. One of our problems, then, is to find the surface content of a spherical simplex in terms of the $\frac{1}{2}n(n-1)$ angles between the n hyperplanes bounding it. This will give us a multihedral angle in terms of its surrounding dihedral angles.

To find the sum S of the dihedral angles in a simplex we choose a point O inside the simplex and drop perpendiculars to the faces. Calling the sum of the $\frac{1}{2}n(n+1)$ angles between these $n+1$ perpendiculars R, $S=\frac{1}{2}n(n+1)\pi-R$. But if we draw a unit sphere with center O, R is the sum of the (spherical) distances between the points $p_1, \cdots, p_{n+1}$ in which these perpendiculars intersect the surface of the sphere. Now, since O is inside the simplex there is no plane through O with all of the perpendiculars lying on one side of it, and hence the points $p_1, \cdots, p_{n+1}$ are not contained in any hemisphere. We call such a set of points *global*.

Our second problem then, discussing the sum of the dihedral angles in a simplex, has been reduced to that of discussing the sum of the distances between the points of a global $(n+1)$-tuple in S_{n-1}.

3. The spherical excess of a spherical simplex. We have to deal with the surface content A of a spherical simplex K in S_{n-1}. We can consider K as the common part of n hemispheres $K_1, \cdots, K_n$, whose boundaries (each an S_{n-2}) we denote by $K_1', \cdots, K_n'$. If we let α_{ij} denote the angle between K_i' and K_j', and A_{ij} denote the content of the spherical lune which is the common part of K_i and K_j, then $\alpha_{ij}=2\pi A_{ij}/C_n$, where $C_n=2\pi^{n/2}/\Gamma(n/2)$, the surface content of the S_{n-1}.

Now taking the origin at the center of the S_{n-1}, the coordinates of the points in K_i satisfy a linear homogeneous inequality $f_i(x)\geqq 0$, and K_i' is given by $f_i(x)=0$. The n forms $f_1, \cdots f_n$ divide the entire sphere into 2^n regions, each region being characterized by the signs of the n functions in it. We shall designate a typical region $R_{i_1,\cdots,i_m}$, meaning that $f_{i_1}, \cdots, f_{i_m}$ are positive in the interior of this region and the other functions negative. By $R^*_{i_1,\cdots,i_m}$ we mean the region diametral to $R_{i_1,\cdots,i_m}$, that is, the region in which $f_{i_1}, \cdots, f_{i_m}$ are negative and the others positive.

With this notation,

$$A_{ij} = A + \sum R_{ij\alpha_i\cdots\alpha_m}, \tag{1}$$

where $R_{ija_1\cdots a_m}$ ranges over all the regions in which f_i and f_j are positive. Now suppose a region R makes p forms positive and $n-p$ negative. Then either R or R^* will occur in $\frac{1}{2}p(p-1)+\frac{1}{2}(n-p)(n-p-1)$ of the equations (1).

Now

$$\frac{p(p-1)}{2}+\frac{(n-p)(n-p-1)}{2}=p^2-np+\frac{n^2-n}{2}$$

and we must consider the maximum and minimum values of this function for integral values of p less than n. Computation shows that the maximum equals $\frac{1}{2}(n-1)(n-2)$ and is taken on for $p=n-1$, and the minimum is attained for $p=[n/2]$. When n is odd, the minimum is $(n-1)^2/4$ and for n even the minimum is $n(n-2)/4$.

Thus adding the $\frac{1}{2}n(n-1)$ equations (1) we will have

$$\sum A_{ij} \leqq \frac{n(n-1)}{2}A+\frac{(n-1)(n-2)}{2}\sum R_\beta,$$

where R_β takes on the value of the surface content of one out of each diametral pair of regions. Thus we have

$$\sum A_{ij} \leqq (n-1)A+\frac{(n-1)(n-2)}{2}\frac{C_n}{2}.$$

Multiplying by $2\pi/C_n$ gives

$$\sum \alpha_{ij} \leqq \frac{(n-1)(n-2)}{2}\pi+\frac{2(n-1)\pi}{C_n}A.$$

Similarly, for n odd,

$$\sum A_{ij} \geqq \frac{n(n-1)}{2}A+\frac{(n-1)^2}{4}\sum R_\beta,$$

and for n even,

$$\sum A_{ij} \geqq \frac{n(n-1)}{2}A+\frac{n(n-2)}{4}\sum R_\beta,$$

giving

$$\sum \alpha_{ij} \geqq \frac{(n-1)^2}{4}\pi+\frac{n^2-1}{2C_n}A,\ n \text{ odd},$$

$$\sum \alpha_{ij} \geqq \frac{n(n-2)}{4}\pi+\frac{n^2}{2C_n}A,\ n \text{ even}.$$

These inequalities constitute the extension of the formula connecting area and spherical excess. It is clear, further, that these bounds on $\sum A_{ij}$, and hence

on $\sum\alpha_{ij}$, are the best possible since any pair $R_{i_1,\cdots,i_m}$ and $R^*_{i_1,\cdots,i_m}$ can be made to assume a fraction of the spherical content arbitrarily close to 1 by properly choosing the simplex K.

4. Lower bound on the distance sum for global sets. We can now use the results of the last section to find a lower bound on the sum of the distances between points of a set which is global, *i.e.*, not contained in a hemisphere. We first confine our attention to global $(n-1)$-tuples in S_{n-1}.

Let $p_1, \cdots, p_{n+1}$ be a global set in S_{n-1}, let $R=\sum p_ip_j$, and let $H_1, \cdots, H_{n+1}$ be the hemispheres with centers at $p_1, \cdots, p_{n+1}$, with $H_1', \cdots, H_{n+1}'$ their bounding S_{n-2}. Then $p_ip_j=\pi-\alpha_{ij}$, where α_{ij} is the angle between H_i' and H_j' as before.

Letting R_i be the sum of the distances between the points $p_1, \cdots, p_{i-1}$, $p_{i+1}, \cdots, p_{n+1}$, we will have $R=(R_1+\cdots+R_{n+1})/(n-1)$. The common part of the hemispheres $H_1, \cdots, H_{i-1}, H_{i+1}, \cdots, H_{n+1}$ is a simplex K_i, whose surface content we may designate A_i and the sum of whose angles we may call $\sum_i$. Then

$$R_i=\frac{n(n-1)}{2}\pi-\sum_i, \quad \text{and} \quad \sum_i \leqq \frac{(n-1)(n-2)}{2}\pi+\frac{2(n-1)\pi}{C_n}A_i.$$

Thus

$$R_i \geqq (n-1)\pi-\frac{2(n-1)\pi}{C_n}A_i$$

and

$$R \geqq (n+1)\pi-\frac{2\pi}{C_n}(A_1+\cdots+A_{n+1}).$$

Now, since the points p_i are global, the simplexes K_i and K_j have no points in common, since any such point would be the center of a hemisphere containing $p_1, \cdots, p_{n+1}$. Furthermore, diametral to K_i is a simplex K_i^* which does not intersect any K_j either, since the points of K_i^* are outside H_e and the points of K_j are inside H_e, $e\neq i, j$. Thus $K_1, \cdots, K_{n+1}, K_1^*, \cdots, K_{n+1}^*$ are mutually exclusive sets and $A_1+\cdots+A_{n+i}\leqq C_n/2$. Consequently $R\geqq n\pi$.

To find a bound on the distance sum for a global set of k points in S_{n-1} with $k>n+1$ we simply observe that we can imbed the S_{n-1} in an S_{k-2} and move the points slightly to make them a global subset of the S_{k-2}.

Thus we have the result: If $p_1, \cdots, p_k$ form a global set in S_{n-1}, $\sum p_ip_j\geqq(k-1)\pi$. That this bound is the best can be seen by taking a point p_1 and $k-1$ points clustered around p_1', its diametral point.

5. Upper bounds for the distance sum. To find an upper bound on the sum of the distances between points $p_1, \cdots, p_k$ of S_{n-1} we use methods very similar to those of Section 3. We take hemispheres $H_1, \cdots, H_k$ with centers at

$p_1, \cdots, p_k$ and let $H_1', \cdots H_k'$ be the bounding S_{n-2}. If A_{ij} is the content of the common part of H_i and H_j, α_{ij} the angle between H_i' and H_j', then as before, $p_ip_j=\pi-\alpha_{ij}$ and $\alpha_{ij}=2\pi A_{ij}/C_n$. Still following the preceding pattern, we associate a linear homogeneous function f_i with each H_i' and classify points of the sphere according to the signs of the functions f_i. Certain combinations of signs may fail to appear but there will be at most 2^k regions and they will come in pairs of diametral regions $R_{i_1,\cdots,i_m}$ and $R'_{i_1,\cdots,i_m}$.

Now $A_{ij}=\sum R_{ija_1,\cdots,a_m}$ where $R_{ija_1,\cdots a_m}$ ranges over all regions in which f_i and f_j are positive. If R is a region in which p of the functions are positive, $k-p$ then will be positive in R' and either R or R' will appear in

$$\frac{p(p-1)}{2}+\frac{(k-p)(k-p-1)}{2}=p^2-kp-\frac{k^2-k}{2}$$

equations for A_{ij}. The minimum of this function is attained for $p=[k/2]$ and is $(k-1)^2/4$ or $k(k-2)/4$ according as k is odd or even.

Thus

$$\sum A_{ij} \geqq \begin{cases} \dfrac{(k-1)^2}{4}\cdot\dfrac{C_n}{2}, & k \text{ odd}, \\ \dfrac{k(k-2)}{4}\cdot\dfrac{C_n}{2} & k \text{ even}, \end{cases}$$

and

$$\sum \alpha_{ij} \geqq \frac{(k-1)^2}{4}\pi \quad \text{for } k \text{ odd}$$

$$\geqq \frac{k(k-2)}{4}\pi \quad \text{for } k \text{ even}.$$

Letting $R=\sum p_ip_j$, $R=k(k-1)\pi/2-\sum\alpha_{ij}$, and thus when k is odd, $R\leqq(k^2-1)\pi/4$, while when k is even, $R\leqq k^2\pi/4$. This can be summarized by saying $R\leqq[k^2/4]\pi$. It will be noted that no use has been made of the dimension of the sphere and hence the maximum can be attained on any sphere by clustering the k points around two diametral points in groups as nearly equal as possible.

6. Summary. In conclusion we might summarize the results obtained, apply them to the angle sums in a simplex in E_n, and point out in what respect the results are incomplete.

First, if A is the surface content of a simplex in the unit sphere S_{n-1}, and $\sum$ the sum of its dihedral angles, we have

$$\text{I:} \quad \frac{(n-1)(n-2)}{2}\pi+\frac{2(n-1)\pi}{C_n}A \geqq \sum \geqq \left[\frac{(n-1)^2}{4}\right]\pi+\left[\frac{n^2}{2}\right]\frac{A}{C_n}.$$

This expresses equally well the relation between a multihedral angle and its surrounding dihedral angles, and hence can be used in conjunction with the formulas below to obtain bounds on the sum of the multihedral angles in a simplex.

Second, if $p_1, \cdots, p_k$ is any set of points on the S_{n-1} and $R = \sum p_i p_j$, then

II: $$R \leqq \left[\frac{k^2}{4}\right]\pi.$$

If, in addition, the points $p_1, \cdots, p_k$ form a global set, then

III: $$R \geqq (k-1)\pi.$$

We can apply these formulas now to give bounds on S, the sum of the dihedral angles in a simplex in E_n.

If n is odd, we have

IV: $$\frac{n(n-1)}{2}\pi \geqq S \geqq \frac{n^2-1}{4}\pi$$

and if n is even,

IVa: $$\frac{n(n-1)}{2}\pi \geqq S \geqq \frac{n^2}{4}\pi.$$

Since each face of a simplex in E_n is itself a simplex in a lower dimensional space, we can find bounds on the sum of the angles of any dimension in a simplex in E_n. However, it is by no means clear that bounds obtained using the foregoing results would be the best possible. This question remains open for investigation.*

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* After this paper was written my attention was called to Coxeter's review of [4] in *Mathematical Reviews*, January, 1954, p. 55. Formulas IV and IVa are a special case of Höhn's inequalities with $k=2$. I have not seen Höhn's thesis but apparently the methods are quite different. Formulas I, II, and III are not connected with Höhn's work, so far as I can tell.

ON THE ACCURACY OF THE ADJOINT METHOD OF DIFFERENTIAL CORRECTIONS

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1. Introduction. Many problems arising in the applied sciences lead to systems of ordinary differential equations, and it is often desirable to determine not only solutions of such systems but also the effect of small perturbations upon these solutions. The calculation of such "differential effects" can be accomplished in a number of ways. There is the "direct" method in which the desired effects are obtained by the comparison of neighboring solutions. Among the "indirect" ways of predicting small effects the "method of adjoints" is much in use, especially in ballistics and related fields.

Now the system of adjoint differential equations involves the computation of certain partial derivatives, which, especially if empirical functions occur, are not always obtainable with great accuracy. The problem discussed here, seemingly not treated anywhere else, is concerned with just what degree of accuracy is required of the partial derivatives in order to guarantee prescribed exactness of the predicted differential effects.

While the cases taken up here are not of the greatest generality they are nevertheless thought to be sufficiently representative to give useful answers.

2. The fundamental formula. Let us consider the system of ordinary differential equations

$$\dot{z} = f(t, z) \qquad z(0) = z_0 \tag{2.1}$$

where

$$z = \begin{bmatrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_n \end{bmatrix}, \quad \text{and} \quad f = \begin{bmatrix} f_1(t, z_1, \cdots, z_n) \\ \cdot \\ \cdot \\ \cdot \\ f_n(t, z_1, \cdots, z_n) \end{bmatrix}$$

represent column matrices of n elements.

Let, similarly,

$$\dot{v} = h(t, v) \qquad v(0) = v_0 \tag{2.2}$$

be a "neighboring" system in the sense that h differs little from f, and v_0 little from z_0. Assuming that unique solutions $z(t) = z(t, z_0)$, $v(t) = v(t, v_0)$ exist and are known, and putting

$$\Delta z(t) = v(t) - z(t)$$
$$\Delta f(t) = h(t, z(t)) - f(t, z(t)),$$

it follows that the $\Delta z(t)$ satisfy the system

$$(\Delta z)^{\cdot} = a\Delta z + \Delta f(t) + b\Delta z + O[(\Delta z)^2], \tag{2.3}$$

with

$$a = (a_{ij}), \qquad a_{ij} = \partial f_i(t, z(t))/\partial z_j$$
$$b = (b_{ij}), \qquad b_{ij} = \partial \Delta f_i(t)/\partial z_j.$$

Accordingly, the variational equations of (2.1) and (2.2) are

$$\dot{\eta} = B\eta + \Delta f, \qquad B = a + b, \qquad \eta(0) = \Delta z(0) \tag{2.4}$$

where

$$\eta = \begin{bmatrix} \eta_1 \\ \cdot \\ \cdot \\ \cdot \\ \eta_n \end{bmatrix},$$

and η_i is the variation of z_i, $i = 1, 2, \cdots, n$.

Now it can be shown that if $z(t)$ satisfies (2.1) and $v(t) = z(t) + \Delta z(t)$ satisfies (2.2), then $\Delta z(t)$ is approximated by the solution of (2.4). In fact [1],

$$\Delta z(t) = \eta(t) + O(M^2),$$

where

$$M = \max [N_1, M(f_1), \cdots, M(f_n)],$$

with

$$N_1 = \max [\|\Delta z(0)\|, \max_t \|\Delta f(t)\|], \qquad \|\Delta z\| = \left[\sum_i (\Delta z_i)^2 \right]^{1/2}$$

and

$$M(f_k) = \max \left[\sum_i (\partial f_k(t, z)/\partial z_j)^2 \right]^{1/2}, \qquad k = 1, 2, \cdots, n.$$

The system adjoint to (2.4) is

$$\dot{\lambda} = -B'\lambda, \tag{2.5}$$

the prime denoting transposition of the matrix.

Since, by (2.4) and (2.5),

$$(\lambda'\eta)^{\cdot} = \lambda'\Delta f,$$

it follows that

$$\lambda'(T)\eta(T) = \lambda'(0)\eta(0) + \int_0^T \lambda'(\tau)\Delta f(\tau)d\tau, \tag{2.6}$$

for any $t = T$.

This is the fundamental formula; it permits the determination of the effect of small perturbations $\eta(0)$ upon quantities q whose differential changes dq at the

value T of the independent variable are expressible in the form $dq=\lambda'(T)\eta(T)$. There are many quantities capable of such representation. In fact, the following general relationship holds: Let u, w be any two variables usable as independent variables along the undisturbed solution near a point P, and let ξ be any disturbance. Let further the differential effect of ξ on any quantity q at equal values u_0 of u be denoted by $dq(\xi \mid u=u_0)$. Then the differential effect of ξ on q at equal values u_0 of u, and of equal values w_0 of w are related by

$$dq(\xi \mid w = w_0) = dq(\xi \mid u = u_0) - (dq/dw)_P dw(\xi \mid u = u_0). \tag{2.7}$$

A detailed proof of this theorem is given in [2]. This proof may be outlined as follows: Let P be the point on the undisturbed solution at which u, w, q assume, respectively, the values u_0, w_0, q_0. For sufficiently small M there exists on the disturbed solution a point Q at which $u=u_0$, and a point R at which $w=w_0$. Then at point Q

$$\begin{aligned} u_Q &= u_0 \\ w_Q &= w_0 + \Delta w(\xi \mid u = u_0) \\ q_Q &= q_0 + \Delta q(\xi \mid u = u_0). \end{aligned}$$

Similarly, at R:

$$\begin{aligned} u_R &= u_0 + \Delta u(\xi \mid w = w_0) \\ w_R &= w_0 \\ q_R &= q_0 + \Delta q(\xi \mid w = w_0). \end{aligned}$$

On the arc QR then the quantity q changes by

$$q_R - q_Q = \Delta q(\xi \mid w = w_0) - \Delta q(\xi \mid u = u_0),$$

the quantity w by

$$w_R - w_Q = -\Delta w(\xi \mid u = u_0).$$

By the mean value theorem there exists a value w^* between w_R and w_Q so that

$$(q_R - q_Q)/(w_R - w_Q) = (dq/dw)_{w^*}. \tag{2.8}$$

On the other hand,

$$(dq/dw)_{w^*} = (dq/dw)_P + O(M).$$

With the proper substitutions into equation (2.8), and using the further facts that

$$\begin{aligned} \Delta q(\xi \mid u) &= O(M) \\ \Delta q(\xi \mid u = u_0) &= dq(\xi \mid u = u_0) + O(M^2), \end{aligned} \tag{2.8a}$$

there is now readily obtained the equation (2.7), as claimed.

Applying (2.7) now to a set of quantities $q=x$, $w=y$, $u=t$ there results

$$dx(\xi \mid y) = dx(\xi \mid t) - (dx/dy)_P dy(\xi \mid t).$$

For the set $q=t$, $w=y$, $u=t$,

$$dt(\xi \mid y) = -(dt/dy)_P dy(\xi \mid t).$$

For $y=0$, $t=T$ the preceding equations become

$$dx(\xi \mid 0) \equiv dR = \eta_1(T) - [\dot{x}(T)/\dot{y}(T)]\eta_2(T) \tag{2.9}$$

$$dt(\xi \mid 0) \equiv dT = -[1/\dot{y}(T)]\eta_2(T). \tag{2.10}$$

It follows then from (2.9) and (2.10) that in general

$$dq = \lambda'(0)\eta(0),$$

where the integration of the adjoint equations (2.5) resulting in a desired effect dR of the variable x at $t=T$ must consequently be carried out starting with the terminal conditions

$$\lambda'(T) = [1, m, 0, \cdots, 0], \tag{2.11}$$

that of the equations (2.5) leading to a desired value dT of the variable t at $t=T$ with the terminal conditions

$$\lambda'(T) = [0, n, 0, \cdots, 0], \tag{2.12}$$

with $m = -[\dot{x}(T)/\dot{y}(T)]$, $n = -[1/\dot{y}(T)]$.

By (2.8) we have

$$\Delta q = \lambda'(0)\eta(0) + O(M^2).$$

Now it is clear that in computing the matrix B certain small errors ϵ are unavoidable, so that the system (2.5) will be replaced actually by one of the form

$$\dot{\Lambda} = -A' \odot \Lambda, \qquad \Lambda(T) = \lambda(T), \tag{2.13}$$

with the symbol $\odot$ denoting pseudo-multiplication.

This, in turn, will lead to a quantity

$$Dq = \Lambda'(0) \odot \eta(0),$$

and thence to a quantity

$$S(\eta) = Dq(\eta) - \Delta q(\eta) - \Delta q(\eta),$$

whose magnitude provides a measure not only of the second order terms but also of the effect of the errors ϵ upon the predicted value dq. Either S or

$$\sigma \equiv [dS/d(\Delta q)]_{\eta=0} = \sum_i [\partial S/\partial \eta_i(0)][\partial \eta_i(0)/\partial \Delta q]_{\eta=0}$$

$$= \sum_i [\Lambda_i(0) - \lambda_i(0)]\lambda_i^{-1}(0) \equiv \sum_i \sigma_i$$

may be used for such a measure, the latter being more useful for comparative purposes. It is clear that σ is the slope at the origin $\eta=0$ of the graph of S versus dq.

3. The case $n=1$. An interesting general result may be obtained easily in the case $n=1$. Then

$$\dot{Z} = F(Z, t), \qquad \dot{\Lambda} = -F_z(Z(t),t) \odot \Lambda.$$

Since $F_z \odot \Lambda = F_z\Lambda + \rho_1$, we have

$$\dot{\Lambda} = -F_z\Lambda - \rho_1,$$

with ρ_1 denoting a small function of t. Thus

$$\Lambda(t) = \exp\left(-\int_0^t F_z d\tau\right)\left[C - \int_0^t \rho_1(\tau) \exp\left(\int_0^t F_z d\tau'\right) d\tau\right].$$

With the terminal condition $\Lambda(T)=1$,

$$\Lambda(t) = \exp\left(-\int F_z d\tau\right)\left[\exp\left(\int_0^T F_z d\tau\right) + \int_0^T \rho_1 \exp\left(\int F_z d\tau'\right) d\tau\right].$$

For $t=0$, then,

$$\Lambda(0) = \exp\left(\int_0^T F_z d\tau\right) + \int_0^T \rho_1 \exp\left(\int F_z d\tau'\right) d\tau. \tag{3.1}$$

Consequently,

$$\Lambda(0) - \lambda(0) = \exp\left(\int_0^T F_z\right) - \exp\left(\int_0^T f_z\right) + \int_0^T \rho_1 \exp\left(\int F_z\right).$$

Let us define now

$$\epsilon(t) = F_z(Z) - f_z(z).$$

Then obviously

$$\Lambda(0) - \lambda(0) \approx \exp\left(\int_0^T f_z d\tau\right)[\exp I(T) - 1] + \int_0^T \rho_1 \exp\left(\int F_z\right),$$

where

$$I(T) = \exp \int_0^T \epsilon(\tau) d\tau. \tag{3.2}$$

Since

$$\begin{aligned} S &= \Lambda'(0) \odot \eta(0) - \lambda(0)\eta(0) \\ &= \rho_2(\eta) + [\Lambda(0) - \lambda(0)]\eta(0), \end{aligned}$$

it follows that

$$\sigma \approx (\exp(I(T)) - 1) + \exp\left(-\int_0^T f_s\right)\left[(\rho_2^*(\eta) + \int_0^T \rho_1 \exp\left(\int f_s\right)\right] \tag{3.3}$$

where ρ_1, ρ_2, and ρ_2^* denote small functions.

We are now especially interested in the effect of ϵ upon σ; putting ρ_1 and ρ_2 identically zero we find

$$\sigma \approx \exp(I(T)) - 1 \approx T\epsilon(t^*), \tag{3.4}$$

if we assume $|T\epsilon(t)| \ll 1$.

The slope σ is thus seen to be approximately proportional to the error ϵ in the partial derivative, as well as the total length T of integration, a result which seems reasonable. If, for example, $T=20$, $f_s=0.1$, $\epsilon \approx 0.001$, then $\sigma \approx 0.02$; errors in f_s of about 1% lead to values in the slope σ of about 2%. On the other hand, in order to expect the slope for a solution integrated to $t=200$ to be about 1%, the partial derivative must be correct to about 0.00005.

4. The analytical example. Let us next consider a special case involving a solution z having four components. It is well known that the plane motion of the center of gravity of a body subject to gravity and resistance may be described in a system x, y of rectangular coordinates by a set of equations of the form (2.1) with

$$z = \begin{bmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{bmatrix}, \quad f = \begin{bmatrix} \dot{x} \\ \dot{y} \\ -k\dot{x} \\ -k\dot{y} - g \end{bmatrix}, \quad z(0) = \begin{bmatrix} x_0 \\ y_0 \\ \dot{x}_0 \\ \dot{y}_0 \end{bmatrix}. \tag{4.1}$$

Here g is the gravitational acceleration, and $k(y, \dot{x}, \dot{y}) = G(\dot{x}, \dot{y})H(y)C^{-1}$, where $H(y)$ is the air density ratio, and G, C are drag function and ballistic coefficient, respectively, functions that are usually determined by experiments. However, in order to obtain closed expressions for the solutions we shall limit ourselves here to the case where k is so slowly changing that it may be considered a positive constant. Then (4.1) is solved by

$$x(t, x_0, \dot{x}_0) = x_0 + \dot{x}_0 k^{-1}(1 - e^{-kt}) \tag{4.2}$$

$$y(t, y_0, \dot{y}_0) = y_0 + (y_0 + \gamma)k^{-1}(1 - e^{-kt}) - \gamma t \tag{4.3}$$

with $\gamma = gk^{-1}$. These expressions obviously tend to the well-known trajectory solutions in vacuum as k approaches zero. Solving (4.2) for $t=t(x, x_0, \dot{x}_0)$, and substituting into (4.3) leads to

$$y(x, x_0, y_0, \dot{x}_0, \dot{y}_0) = y_0 + (\dot{y}_0 + \gamma)(x - x_0)\dot{x}_0^{-1} - \gamma k^{-1} \log\{\dot{x}_0[\dot{x}_0 - k(x - x_0)]^{-1}\}. \tag{4.4}$$

The range R is then defined by means of

$$y(R, 0, 0, \dot{x}_0, \dot{y}_0) = 0, \tag{4.5}$$

and the time of flight T by means of

$$x(T, 0, \dot{x}_0) = R. \tag{4.6}$$

In doing this we associate the "normal" trajectory with

$$z(0) = [0, 0, \dot{x}_0, \dot{y}_0].$$

We note, from (4.2) and (4.3), that

$$\begin{aligned} R &= \dot{x}_0 k^{-1}(1 - e^{-kT}) \\ \dot{x}(T) &= \dot{x}_0 e^{-kT} \\ \dot{y}(T) &= \dot{x}_0 \dot{y}_0 - kR(\dot{y}_0 + \gamma)\dot{x}_0^{-1}. \end{aligned} \tag{4.7}$$

Let us next consider a "neighboring" set of solutions, defined as solutions $w(t)$ of (4.1) associated with $w(0) = z(0) + \eta(0)$, where the components $\eta_i(0)$ of $\eta(0)$ are small. Denoting by dR, dT the differential changes in range and time of flight, we find, by (4.1),

$$dR = - y_x^{-1}[y_{x_0}\eta_1(0) + \cdots + y_{\dot{y}_0}\eta_4(0)] \equiv \alpha'\eta(0),$$

with

$$\alpha' = [\alpha_1, \alpha_2, \alpha_3, \alpha_4],$$

and the partial derivatives $y_x = \partial y/\partial x$, *etc.*, to be evaluated at R, 0, O, $\dot{x}_0$, $\dot{y}_0$. A short calculation shows that

$$\begin{aligned} \alpha_1 &= 1 \\ \alpha_2 &= - \dot{x}_0(\dot{x}_0 - ER)[\dot{x}_0\dot{y}_0 - ER(\dot{y}_0 + \gamma)]^{-1} = - \dot{x}(T)/\dot{y}(T) \\ \alpha_3 &= R\dot{x}_0^{-1} \\ \alpha_4 &= - Rx(T)[\dot{x}_0\dot{y}(T)]^{-1}. \end{aligned} \tag{4.8}$$

For the change dT, by (4.3)

$$\begin{aligned} dT &= y_t^{-1}[y_{v_0}\eta_2(0) + y_{\dot{y}_0}\eta_4(0)] \equiv \beta'\eta(0), \\ \beta' &= [0, \beta_2, 0, \beta_4], \end{aligned}$$

and the partial derivatives y_{v_0}, $y_{\dot{y}_0}$ to be computed at T, 0, $\dot{y}_0$. It is seen that

$$\begin{aligned} \beta_2 &= - [\dot{x}_0\dot{y}_0 - ER(\dot{y}_0 + \gamma)]\dot{x}_0^{-1} = - 1/\dot{y}(T) \\ \beta_4 &= - R[\dot{x}_0\dot{y}(T)]^{-1}. \end{aligned} \tag{4.9}$$

5. The solution of the adjoint system. Let us proceed next to the method of adjoints, as it applies to the system (4.1). It was already pointed out in section 2 that in practice (2.5) is replaced by a system of the form (2.13); in our case we wish to put

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -K & 0 \\ 0 & 0 & 0 & -K \end{bmatrix} \tag{5.1}$$

with $K(t) = k - \epsilon(t)$, and $\epsilon(t)$ assumed to be small in the sense that $\max_t |\epsilon(t)| \ll k$. The associated adjoint system is then, written out in detail,

$$\begin{aligned} \dot{\Lambda}_1 &= 0 \\ \dot{\Lambda}_2 &= 0 \\ \dot{\Lambda}_3 &= -\Lambda_1 + K\Lambda_3 \\ \dot{\Lambda}_4 &= \Lambda_2 + K\Lambda_4. \end{aligned} \tag{5.2}$$

The pertinent terminal conditions are of the form

$$\Lambda'(T) = [\lambda_{q1}, \lambda_{q2}, 0, 0], \tag{5.3}$$

with $\Lambda(T) = \lambda(T)$ as exhibited in (2.11) and (2.12), depending on whether $q = R$ or $q = T$.

The solutions of (5.2), (5.3) are

$$\begin{aligned} \Lambda_1 &= \lambda_{q1} \\ \Lambda_2 &= \lambda_{q2} \\ \Lambda_i(t) &= e^{I(t)}\left[C_i - \lambda_{q,i-2}\int_0^t e^{-I(\tau)}d\tau\right], \qquad i = 3, 4, \end{aligned} \tag{5.4}$$

with

$$I(\tau) = k\tau - \int_0^\tau \epsilon(\tau')d\tau'$$

and the C_i denoting constants of integration. Use of (5.3) shows that

$$C_i = \lambda_{q,i-2}\int_0^T e^{-I(\tau)}d\tau,$$

whence

$$\Lambda_i(t) = e^{I(t)}\lambda_{q,i-2}\int_t^T e^{-I(\tau)}d\tau, \qquad i = 3, 4.$$

We need $\Lambda_i(0)$; obviously

$$\Lambda_i(0) = \lambda_{q,i-2}\int_0^T e^{-I(\tau)}d\tau, \qquad i = 3, 4. \tag{5.5}$$

Let us now consider the case $\epsilon(t) \equiv 0$, assuming, then, that the partial derivatives a_{ij} could be calculated exactly. In this case (5.4) becomes

$$\Lambda_i(0) = \lambda_i(0) = \lambda_{q,i-2}k^{-1}(1 - e^{-kT}), \qquad i = 3, 4. \tag{5.6}$$

i. The case $q=R$. Here $\lambda_{q1}=1$, $\lambda_{q2}=m$ so that, by (5.4), (5.5), and (4.7)

$$\lambda'(0) = [1, m, R\dot{x}_0^{-1}, R\dot{x}_0^{-1}m].$$

Thus $\lambda(0)=\alpha$, as defined in (4.8), and the accuracy of $dR=\lambda'(0)\eta(0)$ is illustrated.

ii. The case $q=T$. Here $\lambda_{q1}=0$, $\lambda_{q2}=n$, so that now

$$\lambda'(0) = [0, n, 0, R\dot{x}_0^{-1}n] = \beta',$$

as seen from (4.9). Thus also $dT=\lambda'(0)\eta(0)$. Since now $\Lambda_i(0)-\lambda_i(0)=0$ for all i, it follows that

$$\sigma = \sum_i [\Lambda_i(0) - \lambda_i(0)]\lambda_i^{-1}(0) = 0.$$

In general, however, $\epsilon(t)\not\equiv 0$, whence $\sigma\neq 0$. Let us now assume that a constant value t^* in $0\leqq t\leqq T$ exists such that $\epsilon(t)\approx\epsilon(t^*)\equiv\epsilon^*$. In that case clearly

$$\int_0^T e^{-I(\tau)}d\tau \approx (k - \epsilon^*)^{-1}[1 - e^{-(k-\epsilon^*)T}],$$

and

$$\Lambda_i(0)/\lambda_i(0) \approx [k/(k - \epsilon^*)][(1 - e^{-(k-\epsilon^*)T}/(1 - e^{-kT})].$$

Therefore,

$$\sigma_i \approx (\epsilon^*/k) + [(1 - e^{\epsilon^*T})/(e^{kT} - 1)].$$

Further if we suppose again that $\max_t |\epsilon(t)|T\ll 1$ then

$$\sigma_i \approx (\epsilon^*/k)[1 - kT/(e^{kT} - 1)],$$

provided

$$\max |\epsilon| \ll \min (k, T^{-1}). \tag{5.7}$$

It follows that

$$\sigma \approx 2^j(\epsilon^*/k)[1 - kT/(e^{kT} - 1)] \tag{5.8}$$

with $j=1$ for $q=R$, and $j=0$ for $q=T$.

The slope is thus again seen to be approximately proportional to both ϵ and T. For example, for $T=20$, $k=0.1$, $\epsilon\approx 0.001$ it is found that $\sigma\approx 0.007$; an error of 1% in the value of k leads to an error of about 0.7% in the slope.

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2. See *l.c.* 1, p. 433.

MATHEMATICAL NOTES

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SOLUTION OF A PROBLEM OF E. M. WRIGHT ON CONVEX FUNCTIONS*

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With R denoting the real number field and f a function on R to R, consider the following two statements about f:

(A) $f(x+\delta)-f(x) \geqq f(y+\delta)-f(y)$ for all $\delta>0$ and $x>y$.

(B) $f(\frac{1}{2}x+\frac{1}{2}y) \leqq \frac{1}{2}f(x)+\frac{1}{2}f(y)$ for all x, y.

It is well known that (A) implies (B), and that for continuous f the two are equivalent. (See [2], for example.) In a recent note [3], Professor E. M. Wright showed that (A) is equivalent to certain other interesting inequalities, and raised a question as to the existence of a function f for which (B) is true but (A) is false. The purpose of this note is to describe such a function†. Needless to say, the argument given leans heavily on the Axiom of Choice.

THEOREM. *There is a function f on R to R such that for all x, $y \in R$ it is true that*

(i) $f(\lambda x+(1-\lambda)y) \leqq \lambda f(x)+(1-\lambda)f(y)$ *for all rational* λ *with* $0 \leqq \lambda \leqq 1$;

(ii) *if* $x \neq y$, *there is a* $\delta>0$ *such that* $f(x+\mu\delta)-f(x)<f(y+\mu\delta)-f(y)$ *for all rational* $\mu>0$.

Proof. Recall first that the line R and the plane R^2 are both of dimension $2^{\aleph_0}$ as vector spaces over the rational field.‡ (This follows, for example, from the lemma on p. 20 of [1].) Thus there is an (additive, rationally homogeneous) isomorphism τ of R onto R^2. For each $x \in R$ let $f(x)=|\tau(x)|^2$, where $|\ |$ is the Euclidean norm in the plane R^2. That (i) is true follows readily from the corresponding property of $|\ |^2$ and the fact that τ is an isomorphism. It remains to establish (ii).

Consider an arbitrary pair x and y of distinct points of R and let $x'=\tau(x)$, $y'=\tau(y)$. Let U be the non-empty open set of all $p \in R^2$ such that $(y', p)>(x', p)$, where $(\ ,\ )$ denotes the inner-product in R^2. Since for all z, $q \in R^2$ it is true that $d|z+tq|^2/dt=2(z, q)+2t|q|^2$, it then follows that $|x'+tp|^2-|x'|^2<|y'+tp|^2-|y'|^2$ for all $p \in U$ and $t>0$. Thus with $\delta=\tau^{-1}(p)$ and μ a positive rational number, it is true that $f(x+\mu\delta)-f(x)<f(y+\mu\delta)-f(y)$. It remains only to show that $\tau^{-1}U$ includes at least one positive number. Suppose not. Then τ^{-1} is an additive real function on R^2 which is bounded above on the non-empty open set

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† Since Wright works with functions on $[0, \infty[$ to R, his question is answered by the restriction to $[0, \infty[$ of the function described in the Theorem.

‡ The Axiom of Choice enters in the proof of this fact.

U, and hence is bounded above on a translate V of U which includes the origin. But then τ^{-1} is bounded on the set $V \cap (-V)$, which is a neighborhood of the origin. Since τ^{-1} is rationally homogeneous, it follows that τ^{-1} is continuous at the origin, and hence by additivity at each point of R^2. Since, however, τ^{-1} maps R^2 biuniquely onto R, it cannot be continuous, and the contradiction completes the proof.

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2. G. H. Hardy, J. E. Littlewood, and G. Pólya, Inequalities, Cambridge, 1934.
3. E. M. Wright, An inequality for convex functions, this MONTHLY, vol. 61, 1954, pp. 620–622.

NOTE ON CONVEX FUNCTIONS

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E. M. Wright in an interesting note [1] on a convex inequality states that it is unknown whether functions exist which satisfy condition (i) below and not condition (ii). It is not difficult to construct such a function, making use of a Hamel basis. (This makes use of the axiom of choice. See [2] or [3].) The conditions are as follows:

(i) $$f\left(\frac{a+b}{2}\right) \leqq \frac{f(a)+f(b)}{2} \qquad \text{for real } a \text{ and } b.$$

(ii) If $a \leqq b$ and $\delta > 0$ then $f(a+\delta) - f(a) \leqq f(b+\delta) - f(b)$.

Let H be a Hamel basis for the real numbers over the rationals. Suppose without loss of generality that 1 and π belong to H. Then each real number x has the unique representation $x = \sum_{h \in H} r_{x,h} \cdot h$, where the $r_{x,h}$ are rational numbers, only a finite number of which are not zero. Let

$$f(x) = \sum_{h \in H} r_{x,h}^2 \qquad \text{for each real } x.$$

It is easy to check that (i) holds. To see that (ii) does not hold, let $a=1$, $b=\pi$, and $\delta=1$. Then $a<b$, $\delta>0$, and $f(a+\delta)-f(a)=4-1>2-1=f(b+\delta)-f(b)$.

The function f may be modified so that (i) is still satisfied; and so that (ii) is satisfied for any preassigned set of values of $\delta>0$ of power less than the continuum, but not for all $\delta>0$.

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ON AN INEQUALITY CONCERNING AN INDEFINITE FORM

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1. Introduction. Let us consider the indefinite form

$$\phi(x) = (x_1^p - x_2^p - \cdots - x_n^p)^{1/p}, \qquad p \geqq 1, \tag{1}$$

for values of the x_1 in the region

$$R: \quad \text{a. } x_i \geqq 0, \qquad \text{b. } x_1 > (x_2^p + x_3^p + \cdots + x_n^p)^{1/p}. \tag{2}$$

Observe that $x+y$ belongs to R whenever x and y separately do. This follows from Minkowski's inequality.

The inequality we shall establish is the reverse triangle inequality,

$$\phi(x + y) \geqq \phi(x) + \phi(y). \tag{3}$$

for x and y both in R. The proof will depend upon a representation for $\phi(x)$ which we shall establish in the next section.

2. Representation for $\phi(x)$. Let us now demonstrate that

$$\phi(x) = \operatorname*{Min}_{S(z)} \sum_{i=1}^{n} x_i z_i, \tag{1}$$

where $S(z)$ is the region defined by the inequalities

$$\text{a. } z_1 \geqq 1, \qquad z_k \geqq 0, \qquad k = 2, \cdots, n.$$
$$\text{b. } (z_2^q + \cdots + z_n^q) \leqq z_1^q - 1, \qquad q = p/(p-1).$$

Proof. Using Holder's inequality on the terms $x_2z_2 + \cdots + x_nz_n$ we see that

$$x_2z_2 + \cdots + x_nz_n \geqq -\left(\sum_{k=2}^{n} x_k^p\right)^{1/p}(z_1^q - 1)^{1/q}, \tag{3}$$

and accordingly that the lower bound of $\sum_{i=1}^{n} x_iz_i$ is attained in $S(z)$. Hence

$$\phi(x) = \operatorname*{Min}_{z_1 \geqq 1}\left[x_1z_1 - \left(\sum_{k=2}^{n} x_k^p\right)^{1/p}(z_1^q - 1)^{1/q}\right]. \tag{4}$$

There is a unique minimum at the point where the derivative with respect to z_1 is set equal to zero. At this point we have

$$\frac{x_1}{\left(\sum_{k=2}^{n} x_k^p\right)^{1/p}} = \frac{z_1^{q-1}}{(z_1^q - 1)^{1-1/q}}. \tag{5}$$

Solving for z_1 and performing the required algebraic substitutions, we obtain the stated expression for $\phi(x)$.

3. Proof of the inequality. Using the foregoing lemma for x and y both in R, we have

$$(1)\qquad \phi(x+y) = \operatorname*{Min}_{S(z)} \sum_{i=1}^{n} (x_i + y_i)z_i = \operatorname*{Min}_{S(z)} \left(\sum_{i=1}^{n} x_i z_i + \sum_{i=1}^{n} y_i z_i \right)$$

$$\geqq \operatorname*{Min}_{S(z)} \left(\sum_{i=1}^{n} x_i z_i \right) + \operatorname*{Min}_{S(z)} \left(\sum_{i=1}^{n} y_i z_i \right)$$

$$\geqq \phi(x) + \phi(y),$$

which is the desired result.

The case $n=4$, $p=2$ seems of particular interest.

UNIFORM CIRCULAR MOTION IS SINGULAR

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What can be said of the motion of a particle P if

$$\begin{cases} P \text{ lies on a circle,} \\ |\text{acceleration of } P| = \text{constant} \neq 0? \end{cases}$$

The partial answer that comes at once to mind is the familiar one involving constant speed. We shall show, however, that *this answer is the singular solution to the problem* and that *the general solution is a type of oscillation along an arbitrary quadrant of the circle.*

Let C be the circle, r its radius, and a the constant numerical value of the acceleration. From the decomposition of the acceleration vector into its tangential and normal components we have

$$(1)\qquad (dv/dt)^2 + v^4/r^2 = a^2.$$

Regarded as an equation in dv/dt, (1) yields as locus of double-points precisely $v^2/r = a$. This locus satisfies (1), is not a particular case of the general solution (9) (below), and corresponds to motion with constant speed. We have thus—pending derivation of (9)—justified the title of this note.

To find the general solution of (1) we substitute vdv/ds for dv/dt and then put $w=v^2$, obtaining

$$(2)\qquad (dw/ds)^2 + 4w^2/r^2 = 4a^2.$$

The general solution of (2) gives

$$(3)\qquad v^2 = ar \cos (2s/r),$$

provided that s is measured from an appropriate point.* If we now introduce a central angle θ so that $s=r\theta$, (3) may be written as

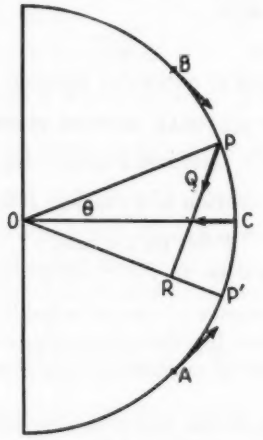
$$(4)\qquad v^2/r = a \cos 2\theta,$$

* The constant of integration here merely corresponds to an arbitrary shifting along C of the path of motion.

which shows that *the motion is confined to the quadrant*

$$-\pi/4 \leqq \theta \leqq \pi/4. \tag{5}$$

Before continuing with the solution, we exploit the fact that (4) suggests a simple method of constructing the acceleration vector geometrically (since v^2/r is precisely the normal component, and a is the length, of that vector). Placing the origin at the center of C and identifying the polar angle with θ, we



draw OP and its reflection (OP') in the polar axis, locate R on OP' (or its extension) so that $OR=PR$, and locate Q on PR (or its extension) in the direction of R from P so that $PQ=a$. Then $\overrightarrow{PQ}$ is the required acceleration vector for the particle P, for: the normal component of $\overrightarrow{PQ}$ is $a\cos 2\theta$ (as required by (4)), since $\angle OPQ=\angle POR=2\theta$ by our construction; $\angle OPQ\leqq\pi/2$ (as required of acceleration vectors in general) by (5); and the tangential component of $\overrightarrow{PQ}$ is directed toward the polar axis (as required by the implication of (4) that $|v|$ decreases as $|\theta|$ increases). (The acceleration vector is shown in the figure also at the ends A, B and the midpoint C of the path of motion.)

Returning to (4) and noting that $v=rd\theta/dt$, we obtain

$$dt = \sqrt{r/a}\,|\,d\theta\,|\,/\sqrt{1-2\sin^2\theta} \quad (t \text{ assumed always increasing}). \tag{6}$$

Next we put

$$\sqrt{2}\sin\theta = \sin\phi, \qquad \theta=\phi=0 \text{ when } t=0,\dagger \tag{7}$$

with ϕ increasing monotonely as θ oscillates between $-\pi/4$ and $\pi/4$. Then from (6) we have‡

† The constant of integration which we anticipate simply determines when we shall take $t=0$.

‡ Standard notations for elliptic integrals and functions may be found in Madelung, Die mathematischen Hilfsmittel des Physikers, Dover, 1943, pp. 76–77.

$$t = \sqrt{r/(2a)} \int_0^{\phi} du/\sqrt{1 - \sin^2 u/2} \equiv \sqrt{r/(2a)} F(1/\sqrt{2}, \phi),$$

whence a period T of the oscillation is given by

$$T = 4\sqrt{r/(2a)} K(1/\sqrt{2})$$

and, by (7) (here we restore the arbitrary constants),

$$s - s_0 = r \arcsin \{(1/\sqrt{2}) \operatorname{sn} [\sqrt{2a/r}(t - t_0)]\}. \tag{8}$$

Finally, differentiation of (8) gives

$$v = \sqrt{ar} \operatorname{cn} [\sqrt{2a/r}(t - t_0)]. \tag{9}$$

ON DIVIDING AN OBJECT EFFICIENTLY

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In what follows E denotes a plane circular disk; "curve" denotes either an arc (homeomorphic image of the closed unit interval) or a simple closed curve (homeomorphic image of the circle). Restricting ourselves to rectifiable curves, we consider some problems closely related to several classical results of the calculus of variations, especially to Dido's problem [1, pp. 3, 446, 465, 488, 528].

Question 1. Let P and Q be two points (not necessarily distinct) on the boundary of a plane circular disk E. What curve in E, joining P to Q and partitioning E into two portions of equal area, has least length?

Solution. If P and Q are diametrically opposite, then the solution is clearly the diameter through P and Q, since the shortest curve joining two points is a straight line and a diameter does in fact divide the disk equally.

Now assume that P and Q are not diametrically opposite one another. Then, by continuity arguments, there is a unique circle C through P and Q such that the area of the portion of the interior of this circle within E is precisely half the area of E. (If P coincides with Q, demand that C be tangent to the boundary of E.)

Now of all curves enclosing a fixed area, it is well known that the circle has minimum length. In particular if we apply this fact to the area contained within C we deduce that the portion of C within E is the curve of minimal length dividing E equally.

Question 2. What curve in E, partitioning E into two connected portions of equal area has least length?

Solution. Any arc in question must join two distinct points P and Q on the boundary of E. Any simple closed curve in question may be translated till it touches the boundary of E in at least one point P. If this arc or curve is to be of minimal length, then it must, by the answer to question 1, be either a circular arc joining P to Q, or a circle tangent at P. Finally, among these arcs and curves,

it is easy to see that the diameter has minimal length.

Question 3. Let R be a rectangle of height h and base $\pi h/2$. Let P be the midpoint of its base. What simple closed curve in E through P, partitioning R into two connected portions of equal area has least length?

Solution. The method which answers question 1 breaks down. The circle given by that method is tangent to the upper base of the rectangle and divides R into three connected portions. There is no simple closed curve of minimal length satisfying the demands of this question.

Further Questions. What are the answers to questions 1 and 2 if E is an ellipse? If a simple closed curve is given on the surface of the sphere, what is the surface of minimal area through this curve dividing the sphere equally (compare to the Plateau Problem)? What if E is to be divided into three equal portions?

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CLASSROOM NOTES

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ROUND ROBIN MATHEMATICS*

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It is quite common that members of a mathematics department are approached by someone in the athletic department requesting help in the construction of a round robin schedule for intramural sports. The purpose of this note is to show a very simple way of constructing round robin schedules for any number of teams.

The requirements for a round robin schedule are that each team plays each other team once and that if there is an even number of teams, each team will play on every day of the "season." If there is an odd number of teams only one team has a "bye" on each day of the season and no team has more than one "bye."

If there are n teams and n *is an odd number* let us number these teams 0,1, 2, $\cdots$, $n-1$. Team i will play against team j on the kth day of the season ($k=0, 1, 2, \cdots, n-1$) if

$$i + j \equiv k \pmod{n}$$

* Work done in connection with Contract No. DA-36-034-ORD-1527RD, Office of Ordnance, U. S. Army.

with the exception that the team for which

$$2i \equiv k \pmod{n}$$

draws a bye. Using the theory of congruences it can easily be shown that this formula meets all of the conditions, *i.e.*, each team plays each other team once, one team gets a "bye" in each round, and each team gets only one "bye." Using this formula we can construct the following table for $n=13$ teams:

k							bye
0	1–12	2–11	3–10	4–9	5–8	6–7	0
1	0–1	2–12	3–11	4–10	5–9	6–8	7
2	0–2	3–12	4–11	5–10	6–9	7–8	1
3	0–3	1–2	4–12	5–11	6–10	7–9	8
4	0–4	1–3	5–12	6–11	7–10	8–9	2
5	0–5	1–4	2–3	6–12	7–11	8–10	9
6	0–6	1–5	2–4	7–12	8–11	9–10	3
7	0–7	1–6	2–5	3–4	8–12	9–11	10
8	0–8	1–7	2–6	3–5	9–12	10–11	4
9	0–9	1–8	2–7	3–6	4–5	10–12	11
10	0–10	1–9	2–8	3–7	4–6	11–12	5
11	0–11	1–10	2–9	3–8	4–7	5–6	12
12	0–12	1–11	2–10	3–9	4–8	5–7	6

If there is *an even number of teams*, say, $n+1$ where n is again an odd number, these teams will be labeled $0, 1, 2, \cdots, n$. Team i will then play against team j on the kth day of the season $(i, j, k=0, 1, 2, \cdots, n-1)$ if

$$i + j \equiv k \pmod{n}$$

with the exception that the team for which

$$2i \equiv k \pmod{n}$$

will play against team n on the kth day of the season. In other words, the team which draws the "bye" in the schedule for $n-1$ teams is matched against team n.

In order to construct a round robin schedule for 14 teams we simply take the

round robin schedule for the 13 teams 0, 1, 2, $\cdots$, 12 and match on each day the team having the bye against team 13.

In some sports it is desirable to designate in each game one team as the "home team." If there is an odd number of teams each team will play an even number of games and it will be the "home team" half of the time if we use the following scheme:

(i) if $i+j$ is *odd*, the home team is the team having the *smaller* number, (ordering the numbers from 0 to $n-1$ in the usual sense).

(ii) if $i+j$ is *even*, the home team is the team having the *larger* number.

For example, in the round robin schedule for 13 teams, team 1 is the home team when $k=0$, 3, 5, 7, 9, and 11 while team 8 is the home team when $k=1$, 4, 6, 8, 10, and 12. It can be shown that this produces the desired results. Of course, if there is an even number of teams, each team plays an odd number of games and no such arrangement is possible.

A GRAPHICAL INTEGRATION

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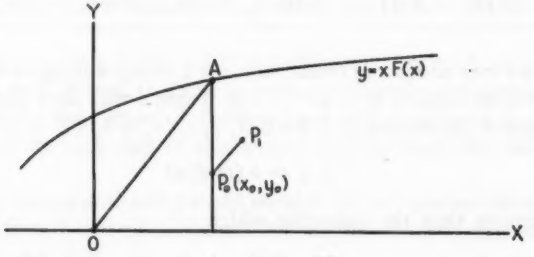
The following is believed to be a new simple graphical method of integrating. Let the differential equation be given by

$$\frac{dy}{dx} = F(x) \qquad \text{(where } x = x_0,\ y = y_0\text{)}. \tag{1}$$

One first plots the curve $y=xF(x)$ (assumed to be continuous). From the point $P_0(x_0, y_0)$, a vertical line is drawn intersecting the curve $y=xF(x)$ in the point A. Then the tangent at P_0 to the integral curve of equation (1) must be parallel to OA. A small segment P_0P_1 of this tangent is drawn with a parallel ruler. From point P_1, the procedure is repeated. The proximity of the successive points P_0, P_1, *etc.* will determine the accuracy of the construction.



This method leads to a particularly simple construction for $y=\log x$.

Remarks: The above method can also be adapted to graphically solving the D.E., $dy/dx=F(x)/G(y)$, $(x=x_0, y=y_0)$. We let $dz/dx=F(x)$, $(x=x_0, z=z_0)$, and $dz/dy=G(y)$, $(y=y_0, z=z_0)$. Both of these latter equations are solved by the above

method yielding the graphs of $z-z_0=\int_{x_0}^{x}F(x)dx$, and $z-z_0=\int_{y_0}^{y}G(y)dy$. It is now a simple matter to obtain the plot of y vs. x. Another method would be to specialize the functions in the graphical solution of

$$\frac{dy}{dx}=\frac{\phi(x)-G(y)}{\psi(x)-F(y)},$$

by letting $G(y)=\psi(x)=0$. (See author's note, this MONTHLY, vol. 61, 1954, pp. 565–567).

ON A GRAPHICAL SOLUTION

M. E. LEVENSON, Brooklyn College

The present paper is concerned with a graphical solution of the equation

$$\frac{dy}{dx}+p(x)y=q(x), \qquad p(x)\not\equiv 0.$$

This construction is based on the following property of this equation. If the family of integral curves of the equation is cut by the line $x=x_1$, the tangents at the points of intersection are concurrent and meet at the point

$$\left(x_1+\frac{1}{p(x_1)},\ \frac{q(x_1)}{p(x_1)}\right).$$

This property of the differential equation recently appeared as a problem on the William Lowell Putnam Examination (March, 1954) and also appears as an exercise in Langer's excellent textbook *A First Course in Ordinary Differential Equations* (p. 67).

The construction is as follows: Draw the curves (see Figure).

$$C_1:\ y=\frac{q(x)}{p(x)}$$

and

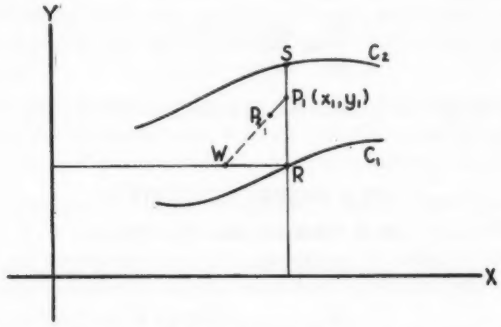
$$C_2:\ y=x+\frac{1}{p(x)}.$$

Consider a point $P_1(x_1, y_1)$. Draw the vertical line through this point meeting C_1 in R and C_2 in S. Draw the horizontal line through R. On the horizontal line, locate the point W whose abscissa is equal to the ordinate of the point S. The point W so constructed has the coordinates

$$\left(x_1+\frac{1}{p(x_1)},\ \frac{q(x_1)}{p(x_1)}\right)$$

and therefore the line joining W to $P_1(x_1, y_1)$ is the tangent line through the

point $P_1(x_1, y_1)$. Now taking a point P_2 on WP_1 close to P_1, we repeat the construction for P_2, *etc.*



It is obvious that the roles of x and y may be interchanged and that the construction can be modified rather simply to give the orthogonal trajectories. It should be mentioned that the graphical procedure given by M. S. Klamkin (this MONTHLY. vol. 61, 1954, p. 565–567) appears to be limited to the cases where his functions F and G have unique inverses.

THE FORMAL SOLUTION OF A DIFFERENTIAL EQUATION

R. K. RITT, University of Michigan

We consider the ordinary differential equation with constant coefficients

$$V_m\left(\frac{d}{dt}\right)x = F(t), \qquad F(t + 2\pi) = F(t) \tag{1}$$

where

$$V_m(p) = \sum_{r=0}^{m} a_{m-r}p^r = a_0\prod_{j=1}^{m}(p - b_j),\ b_i \neq b_j \text{ if } i \neq j,\ b_j \neq \text{an integer.} \tag{2}$$

If

$$[V_m(p)]^{-1} = \sum_{j=1}^{m} A_j(p - b_j)^{-1}, \tag{3}$$

then the formal application of the finite Fourier transform gives

$$\begin{cases} x(t) = \sum_{j=1}^{m} A_j u_j(t) \\ u_j(t) = (1 - \exp 2\pi b_j)^{-1}e^{b_j t} * F(t) \end{cases} \tag{4}$$

where $f(t) * g(t) = \int_0^t f(t-s)g(s)ds + \int_t^{2\pi} f(t+2\pi-s)g(s)ds$. In this note we prove that if $F(t)$ is piecewise continuous, (4) represents a function which has $(m-1)$ con-

tinuous derivatives, and which satisfies (1) on every interval in which $F(t)$ is continuous. The proof is elementary, and easily adapted to a first course in Fourier series.

The proof depends upon the following statement about partial fraction expansions:

If $V_m(p)$ *and* $[V_m(p)]^{-1}$ *are as in* (2) *and* (3), *then*

$$\sum_{j=1}^{m} A_j b_j^r = \begin{cases} 0 & r = 0, \cdots, m-2 \\ a_0^{-1} & r = m-1. \end{cases} \tag{5}$$

This statement is easily verified for $m=1, 2$. We assume it is true for $m=k-1$, $k \geqq 3$ and proceed by induction. Let $V_k(p)=(p-b_k)V_{k-1}(p)$. Then

$$[V_k(p)]^{-1} = (p-b_k)^{-1}[V_{k-1}(p)]^{-1} = (p-b_k)^{-1}\sum_{j=1}^{k-1} B_j(p-b_j)^{-1} = \sum_{j=1}^{k} A_j(p-b_j)^{-1},$$

where

$$\begin{cases} A_j = -B_j(b_k-b_j)^{-1}, & j = 1, \cdots, k-1, \\ A_k = \sum_{j=1}^{k-1} B_j(b_k-b_j)^{-1}. \end{cases} \tag{6}$$

Then, using (6),

$$\sum_{j=1}^{k} A_j b_j^r = \sum_{j=1}^{k-1} B_j(b_k^r - b_j^r)(b_k-b_j)^{-1} = \sum_{s=0}^{r-2} b_k^{r-1-s}\sum_{j=1}^{k-1} B_j b_j^s + \sum_{j=1}^{k-1} B_j b_j^{r-1}. \tag{7}$$

The hypothesis of induction applied to (7) proves (5) for $m=k$.

Now, to continue the main proof, it is easy to verify that the $u_j(t)$ are continuous and that

$$\frac{d}{dt}u_j = b_j u_j + F(t). \tag{8}$$

Making use of (4), (5), and (8), we obtain

$$\left(\frac{d}{dt}\right)^r x = \begin{cases} \sum_{j=1}^{m} A_j b_j^r u_j(t), & r = 0, \cdots, m-1, \\ \sum_{j=1}^{m} A_j b_j^m u_j(t) + a_0^{-1}F(t), & r = m. \end{cases} \tag{9}$$

But from (9), we have

$$V_m\left(\frac{d}{dt}\right)x = \sum_{j=1}^{m} A_j u_j(t) V_m(b_j) + F(t) = F(t), \tag{10}$$

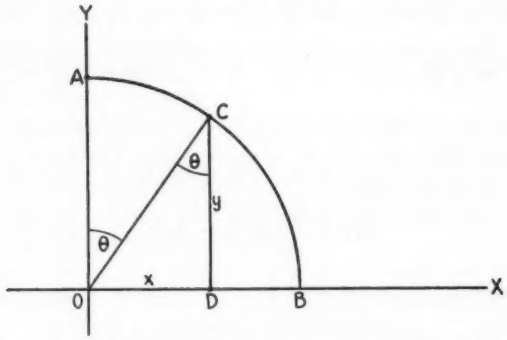
which completes the proof.

ON THE DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

M. R. SPIEGEL, Rensselaer Polytechnic Institute

In the usual course in calculus it is the custom to introduce the trigonometric functions and their derivatives before considering the derivatives of the inverse trigonometric functions. In the so-called unified courses in calculus where some of the elementary concepts of integration are taken up early, there seems to be some advantage in introducing these derivatives in the reverse manner.

To illustrate the possibilities in this direction we show how to find the derivative of arcsin x. Referring to the figure, ACB represents one quarter of



the circumference of a circle with center at the origin and having unit radius. The area of $OACD$ given by

$$\int_0^x \sqrt{1-x^2}\,dx$$

is also equal to the sum of the areas of OCD and OAC. This sum is given by

$$\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\arcsin x$$

from elementary formulas for the area of a triangle and sector of a circle. Hence

$$\int_0^x \sqrt{1-x^2}\,dx=\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\arcsin x. \tag{1}$$

By differentiation it follows that

$$\sqrt{1-x^2}=\frac{d}{dx}\left(\frac{x}{2}\sqrt{1-x^2}\right)+\frac{1}{2}\frac{d}{dx}(\arcsin x)$$

which yields easily enough

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}} \tag{2}$$

or, if one likes,

$$\frac{d}{dx}(\arcsin u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \tag{3}$$

where u depends on x.

From this result it is easy to obtain the derivatives of the other inverse trigonometric functions. Thus

$$\frac{d}{dx}(\arccos x) = \frac{d}{dx}\left(\frac{\pi}{2} - \arcsin x\right) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arctan x) = \frac{d}{dx}\left(\arcsin \frac{x}{\sqrt{1-x^2}}\right) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\operatorname{arc\,cot} x) = \frac{d}{dx}\left(\frac{\pi}{2} - \arctan x\right) = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}(\operatorname{arc\,csc} x) = \frac{d}{dx}\left(\arcsin \frac{1}{x}\right) = \frac{-1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\operatorname{arc\,sec} x) = \frac{d}{dx}\left(\frac{\pi}{2} - \operatorname{arc\,csc} x\right) = \frac{1}{x\sqrt{x^2-1}}.$$

We may now obtain the derivatives of the trigonometric functions. The general procedure is illustrated by determining $d(\sin x)/dx$. Let

$$y = \sin x$$

so that

$$x = \arcsin y.$$

Then it follows at once that

$$\frac{dx}{dy} = \frac{1}{\sqrt{1-y^2}}$$

or

$$\frac{dy}{dx} = \sqrt{1-y^2} = \cos x. \tag{4}$$

Derivatives of the other trigonometric functions can be obtained in a similar way.

The writer believes that the mode of presentation of this paper is more easily

understood by the student than the conventional procedure for several reasons. First, there is no need for the investigation of $\lim_{\theta \to 0} (\sin \theta/\theta)$ which is involved in the conventional procedure. Second, there is no need to manipulate a difference quotient into a form suitable for limiting processes. Third, it is tied in with an interesting geometric interpretation. Incidentally the integration formula (1) is a useful by-product which normally would involve for its evaluation the techniques of integration by parts or trigonometric substitution.

It should be remarked that in obtaining all of the above results, angles were restricted to those lying in the first quadrant. It is not at all difficult to consider other angles, the procedure being entirely analogous to that used in the conventional approach.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1201. *Proposed by C. S. Ogilvy, Hamilton College*

What is the area of the maximum cross section of the unit cube?

E 1202. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let O be an arbitrary point on an arbitrary line λ passing through the centroid G of a tetrahedron $ABCD$. If λ cuts the planes BCD, CDA, DAB, ABC in A', B', C', D', show that

$$A'O/A'G + B'O/B'G + C'O/C'G + D'O/D'G = 4.$$

E 1203. *Proposed by S. I. Birnbaum, Polytechnic Institute of Brooklyn*

Prove that for given sets X, Y, T we have $X = Y = 0$ if and only if

$$T = (X \cap T') \cup (Y \cap T') \cup (X' \cap T \cap Y').$$

E 1204. *Proposed by A. J. Goldman, Princeton University*

Let C be the circle $x^2+y^2=1$, $f(x, y)$ a continuous real-valued function defined on C, and A an angle such that $0 < A \leqq \pi$. Show that there are two radii of C which form an angle A and have endpoints at which $f(x, y)$ has the same value.

E 1205. *Proposed by H. J. Cohen, City College of New York*

If $f''(x)$ is continuous on $[0, 1]$ and if $f(0)=f(1)=0$, show that

$$\int_0^1 |f''(x)/f(x)|\,dx > 4.$$

SOLUTIONS

Mersenne Numbers

E 1171 [1955, 442]. *Proposed by N. D. Brenner, University of Pennsylvania*

(1) Is a number of the form 2^p-1, where p is a prime, ever divisible by another number of the same form?

(2) Are two such numbers necessarily relatively prime?

Solution by W. E. Briggs, University of Colorado. Let h be the smallest positive integer such that $2^h \equiv 1 \pmod{r}$, r a prime. Then, if $2^t \equiv 1 \pmod{r}$, $h \mid t$, since otherwise $2^t = 2^{kh+n} \equiv 2^n \equiv 1 \pmod{r}$, where $0<n<h$. Hence $2^p-1 \equiv 2^q-1 \equiv 0 \pmod{r}$, r a prime, implies that h divides both p and q.

Also solved by H. L. Alder, G. E. Bardwell, W. J. Blundon, I. A. Dodes, D. C. Duncan, A. D. Freedman, A. J. Goldman, Virginia S. Hanly, Vern Hoggatt, Douglas Holdridge, A. R. Hyde, D. C. B. Marsh, C. S. Ogilvy, G. B. Robison, Azriel Rosenfeld, R. R. Seeber, Jr., R. E. Shafer, Allen Simon, Donald Solitar, E. P. Starke, R. P. Tapscott, and the Proposer. Late solutions by Hüseyin Demir and R. K. Meany.

Editorial Note. We may show more generally that if n is any positive integer greater than 1, then $(n^a-1, n^b-1)=1$ if and only if $(a, b)=1$.

A Partitioning of the Binomial Coefficients

E 1172 [1955, 442]. *Proposed by D. R. Morrison, Tulane University*

Show, for every positive integer n, that of the following sums of binomial coefficients two are equal and the third differs from them by 1; that is, they partition their sum, 2^n, into three parts which are as nearly equal as three integers whose sum is 2^n can be.

$$S_1 = \binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \cdots$$

$$S_2 = \binom{n}{1} + \binom{n}{4} + \binom{n}{7} + \cdots$$

$$S_3 = \binom{n}{2} + \binom{n}{5} + \binom{n}{8} + \cdots$$

I. *Solution by Patricia Ann Brown, Trinity University, Texas.* Let ω be a root of $x^2+x+1=0$. Then

$$\begin{aligned} 1 &= (1+\omega)(1+\omega^2) = (1+\omega)^n(1+\omega^2)^n \\ &= (S_1+\omega S_2+\omega^2 S_3)(S_1+\omega^2 S_2+\omega S_3) \\ &= [(S_1-S_2)^2+(S_2-S_3)^2+(S_3-S_1)^2]/2. \end{aligned}$$

II. *Solution by Virginia S. Hanly, Ohio State University.* Using the identity

$$\binom{n}{k}+\binom{n}{k+1}=\binom{n+1}{k+1}$$

we readily find that

$$\begin{aligned} S_1(n)+S_2(n) &= S_2(n+1), \\ S_1(n)+S_3(n) &= S_1(n+1), \\ S_2(n)+S_3(n) &= S_3(n+1). \end{aligned}$$

Clearly, if the desired result is true for any number n, it is also true for the succeeding number $n+1$. It is easily verified to be true for $n=2$. Hence the desired result is true for all n by mathematical induction.

III. *Note by E. P. Starke, Rutgers University.* This is exactly problem E 300 [1938, 320], proposed by Daniel Finkel and solved by Emma Lehmer. In the *National Mathematics Magazine*, March 1939, pp. 292–293, the following results are given:

If the binomial coefficients are distributed analogously into four sets, two of the sums are always equal and the other two are equal for odd n but unequal for even n; the differences between the sums are always powers of 2.

If the binomial coefficients are distributed analogously into five sets, the five sums will have just three distinct values; the three possible differences of distinct sums being three consecutive terms of the Fibonacci series,

$$1, 1, 2, 3, 5, 8, 13, \cdots, a_{n+1} = a_n + a_{n-1}.$$

The property of equality of certain sums holds for any number of sets, but the facts about the differences of sums do not continue to work out so neatly.

Also solved by G. E. Bardwell, W. J. Blundon, Louis Brickman, J. R. Byrne, Leonard Carlitz, G. B. Charlesworth, Richard Courter, V. D. Gokhale, B. K. Gold, A. J. Goldman, C. D. Gorman, Douglas Holdridge, A. R. Hyde, P. G. Kirmser, M. S. Klamkin, D. C. B. Marsh, C. S. Ogilvy, C. D. Olds, Paul Payette, Walter Penney, L. A. Ringenberg, H. A. Robinson, Azriel Rosenfeld, R. R. Seeber, Jr., R. E. Shafer, R. P. Tapscott, and Chih-yi Wang. Late solutions by Hüseyin Demir and W. S. Lawton.

Klamkin called attention to the similar problem No. 117, *Mathematics Magazine*, Nov.–Dec. 1952, p. 104.

Computation of an Arc Length

E 1173 [1955, 442]. *Proposed by R. A. Laird, New Orleans, La.*

The sum of the lengths of five equal contiguous chords inscribed in a given circular arc is 5280 feet. The length of the long chord of the given arc is 5208 feet. Find, to the nearest inch, the length of the given arc.

Solution by A. R. Hyde, West Hartford, Conn. Let R, S, $10A$ denote the radius, arc length, and central angle respectively. Then

$$\sin A = 528/R, \qquad \sin 5A = 2604/R = (217/44)\sin A,$$

$$S = 10RA = 5280(A/\sin A) = 5280[1 + (\sin^2 A)/6 + 3(\sin^4 A)/40 + \cdots].$$

But

$$\sin 5A = 5 \sin A - 20 \sin^3 A + 16 \sin^5 A,$$

whence

$$16 \sin^4 A - 20 \sin^2 A + 3/44 = 0$$

and

$$\sin^2 A = 5/8 - (1/2)\sqrt{17/11} = 0.00342.$$

Two terms of the series suffice, giving

$$S = 5280(1.00057) \text{ feet} = 5283 \text{ feet to the nearest inch.}$$

Also solved by W. V. Gamzon, Raymond Huck, Edgar Karst, Sam Kravitz, D. C. B. Marsh, Morris Morduchow, C. S. Ogilvy, Walter Penney, N. C. Perry, L. A. Ringenberg, Azriel Rosenfeld, R. R. Seeber, Jr., Chih-yi Wang, R. H. Wilson, Jr., and the Proposer. Several incorrect solutions were also received. Late solution by L. V. Mead and Don Cenis (jointly).

The Proposer has developed the approximation formula

$$S = P + (P - C)/(n^2 - 1), \quad \text{with error} \quad E < S^5/1920n^2R^4,$$

where

S = required arc,
C = long chord of the arc,
n = number of equal contiguous chords inscribed in the arc,
P = sum of the equal contiguous chords,
R = radius of the arc.

The case $n = 2$ of this approximation formula is the well known Huygens' formula for the approximation to the length of a circular arc.

Hermitian Matrices

E 1174 [1955, 442]. *Proposed by M. D. Marcus and B. N. Moyls, University of British Columbia*

Let A be an n-square Hermitian matrix whose characteristic roots are the diagonal elements A_{ii}, $i=1, \cdots, n$. Prove that A is diagonal.

Solution by W. V. Parker, Alabama Polytechnic Institute. If the characteristic roots of $A=(A_{ij})$ are $A_{11}, A_{22}, \cdots, A_{nn}$, then

$$\sum_{i<j} A_{ii}A_{jj} = \sum_{i<j} (A_{ii}A_{jj} - A_{ij}A_{ji}).$$

This implies

$$\sum_{i<j} A_{ij}A_{ji} = 0.$$

In case A is Hermitian, this becomes

$$\sum_{i<j} A_{ij}\overline{A}_{ij} = 0,$$

which is possible only if $A_{ij}=0$, $i \neq j$. For a generalization, see this MONTHLY [1954, 330].

Also solved by G. E. Bardwell, Leonard Carlitz, C. D. Gorman, D. S. Greenstein, M. S. Klamkin, D. C. B. Marsh, T. F. Mulcrone, and the Proposers.

Mutually Tangent Spheres

E 1175 [1955, 442]. *Proposed by G. A. Yanosik, New York University*

(1) Three mutually tangent spheres, with radii $r_1<r_2<r_3$, rest upon a horizontal plane. Find the radius R of the largest sphere which will slip through the space between the three given spheres.

(2) Given four mutually tangent spheres of slightly different radii $r_1<r_2<r_3<r_4$, find the radius R of the largest fifth sphere which will fit in the space which is more or less bounded by the four given spheres.

Notes by Leon Bankoff, Los Angeles, Calif. (1) When the fourth sphere is in the critical position for passage, the centers of the four spheres are coplanar. The problem is thus reduced to its two-dimensional analogue, having the solution (see *School Science and Mathematics*, Jan. 1953, p. 75, problem 2293).

$$R = r_1r_2r_3/(\sqrt{2r_1r_2r_3(r_1+r_2+r_3)} + r_1r_2 + r_2r_3 + r_3r_1).$$

The initial condition of tangency of the three given spheres to a horizontal plane merely implies that $r_1 \geqq r_2r_3/(\sqrt{r_2}+\sqrt{r_3})^2$, and has no bearing on the general solution. See this MONTHLY [1941, 267], problem E 432.

(2) A fifth sphere will fit in the space "bounded" by the four given spheres

only if such space exists. The condition $r_1 < r_2 < r_3 < r_4$ is not sufficient to preclude the possibility that the centers of the four given mutually tangent spheres may be coplanar. Clearly, from the preceding solution, we must impose the restriction

$$r_1 > r_2r_3r_4/(\sqrt{2r_2r_3r_4(r_2 + r_3 + r_4)} + r_2r_3 + r_3r_4 + r_4r_2).$$

A solution to this problem was given by Professor Gill in *Lady's and Gentleman's Diary*, 1884, p. 66. The final answer can be given in various forms. For example (see N. A. Court, *Modern Pure Solid Geometry*, Ex. 21, p. 210),

$$1/R^2 - (1/R)\sum(1/r_1) + \sum(1/r_1^2) - \sum(1/r_1r_2) = 0,$$

or (Professor F. Soddy's form, given in *Nature*, vol. 137, p. 1021, June 20, 1936),

$$[\sum(1/r_1) + 1/R]^2 = 3[\sum(1/r_1^2) + 1/R^2].$$

If R_0 is the radius of the sphere circumscribing the four given spheres, then it is interesting that

$$1/R = \sum(1/r_1) + 1/R_0.$$

Also solved by W. B. Carver, D. C. B. Marsh and the Proposer.

The Proposer showed that for part (2)

$$R = 2r_1r_2r_3r_4/[(\sum r_1r_2r_3) + \sqrt{12r_1r_2r_3r_4(\sum r_1r_2) - 3(\sum r_1r_2r_3)^2}].$$

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4673. *Proposed by Paul Erdös, Haifa, Israel.*

Let $a_1 < a_2 < \cdots$; $b_1 < b_2 < \cdots$ be two sequences of integers. Prove that from the sequence $a_i + b_j$ one can always select an infinite subsequence such that no one divides another.

4674. *Proposed by R. Venkatachalam Iyer, Karamana, Trivandrum, India*

The general solution in rational numbers of the equation

$$x^2 + y^2 + z^2 + 2xyz = 1 \tag{1}$$

is given by I. A. Barnett as

$$x = \frac{b^2 + c^2 - a^2}{2bc}, \qquad y = \frac{c^2 + a^2 - b^2}{2ca}, \qquad z = \frac{a^2 + b^2 - c^2}{2ab}.$$

See *A Diophantine Equation Characterizing the Law of Cosines*, this MONTHLY, April 1955, p. 251. Find the general solution of (1) in integers.

4675. *Proposed by D. H. Lehmer, University of California at Berkeley*

Show that

$$\left(\frac{3\cdot 3}{1\cdot 5}\right)^4 \left(\frac{15\cdot 15}{13\cdot 17}\right)^4 \left(\frac{27\cdot 27}{25\cdot 29}\right)^4 \left(\frac{39\cdot 39}{37\cdot 41}\right)^4 \left(\frac{51\cdot 51}{49\cdot 53}\right)^4 \cdots = 12.$$

4676. *Proposed by J. E. Wilkins, Jr., Nuclear Development Corporation of America, White Plains, N. Y.*

Let $f(x)$ be a real valued measurable function defined on a measurable set A with measure μ such that

$$0 < a \leqq f(x) \leqq b < \infty$$

almost everywhere on A. Let $\bar{f}$ be the average value of f and $\tilde{f}$ be the reciprocal of the average value of the reciprocal of f. Find least upper and greatest lower bounds for $\bar{f}/\tilde{f}$.

4677. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn, N. Y.*

For what values of θ does the following series converge:

$$\sum_{n=1}^{\infty} \frac{1}{n \sin 2^n\theta}\,?$$

SOLUTIONS

A Finite Sum

4211 [1946, 397]. *Proposed by K. L. Chung, Syracuse University*

Prove that

$$\binom{nd}{n} = d(d-1) \sum_{k=0}^{n} \frac{(dk-2)!}{(k-1)!(dk-k)!} \binom{nd-kd}{n-k}.$$

Editorial Note. The equation of this problem occurs as (13) in an article by H. W. Gould, *Generalizations of Vandermonde's Convolution*, this MONTHLY,

this issue, p. 88. For a complete proof and for other similar and related sums the article should be consulted.

A Function Zero Almost Everywhere

4613 [1954, 718]. *Proposed by Joseph Lehner and G. M. Wing, Los Alamos Scientific Laboratory*

Let $\phi(t)$ be integrable over (0, 1) and suppose, for $z=re^{i\theta}$, that

$$F(z) = \int_0^1 e^{\alpha z t}\phi(t)dt, \qquad R\alpha > 0,$$

satisfies

$$\int_0^\infty |F(r)|^2 dr < \infty.$$

Show that $\phi=0$ almost everywhere.

Solution by R. P. Boas, Jr., Northwestern University. If $\phi(t)$ is not almost everywhere zero, it is intuitive that $F(z)$ must grow exponentially in any direction such that $R(\alpha z)>0$, and hence $\int_0^\infty |F(r)|^2 dr = \infty$. This can be made rigorous as follows. $F(z)$ is an entire function of exponential type, bounded for $\arg z = -\arg \alpha \pm \pi/2$. For such a function, a theorem of Ahlfors and Heins states that $r^{-1}\log|F(r)| \to c$ as $r\to\infty$, except for a set E of values of r such that $\int_E r^{-1}dr$ converges; and from a theorem of Paley and Wiener it follows that $c=\lambda|\alpha|\cos(\arg\alpha)$, where λ is the smallest number such that $\phi(z)=0$ almost everywhere in $(\lambda, 1)$. (*Cf.* Boas, *Entire Functions*, 1954, pp. 108, 116.) If $\lambda=0$, $F(z)\equiv 0$ and (by the inversion formula for Fourier or Laplace transforms) we are through. If $\lambda>0$, we have $|F(r)|>e^{br}$, $0<b<c$, for $r>R$, say, except for the set E. If H is the complement of E (for $r>R$) we have

$$\infty > \int_H |F(r)|^2 dr > \int_H e^{2br}dr > \int_H r^{-1}dr,$$

which is impossible since $\int_{E+H} r^{-1}dr = \infty$. Evidently the condition $\int_0^\infty |F(r)|^2 dr < \infty$ can be replaced by much less restrictive ones.

Also solved by the Proposers.

Definite Integral of an Infinite Product

4614 [1954, 718]. *Proposed by H. F. Sandham, Institute for Advanced Studies, Ireland*

Prove that

$$\int_0^1 (1-x)(1-x^2)(1-x^3)\cdots dx = \pi\sqrt{\frac{48}{23}}\,\frac{\sinh \frac{1}{2}\pi\sqrt{23}}{\cosh \frac{1}{2}\pi\sqrt{23}}.$$

Solution by T. F. Mulcrone, S.J., St. Charles College, Grand Coteau, La. The termwise integration of Euler's expansion

$$x^{1/24}(1-x)(1-x^2)(1-x^3)\cdots = x^{1^2/24} - x^{5^2/24} - x^{7^2/24} + x^{11^2/24} + \cdots$$

leads, as the Proposer shows in his Note (this MONTHLY [1954, 104]), to the definite integral

$$\int_0^1 x^{\alpha+1/24}(1-x)(1-x^2)(1-x^3)\cdots\frac{dx}{x} = \frac{\pi\sqrt{2}\sinh\tfrac{1}{2}\pi\sqrt{24\alpha}}{\sqrt{\alpha}\cosh\tfrac{1}{2}\pi\sqrt{24\alpha}}$$

($\alpha>1/24$). When $\alpha=23/24$, we have the desired result.

Also solved by W. J. Blundon, Armand Brumer, Leonard Carlitz, Oscar Goldman, C. D. Olds, M. R. Spiegel, T. W. Summers, Chih-yi Wang, and the Proposer. Late solution by G. B. Findley.

A Trigonometric Limit

4615 [1954, 718]. *Proposed by D. J. Newman, Republic Aviation Corporation, Farmingdale, N. Y.*

Define

$$F(x) = \cos x \cos\frac{x}{2}\cos\frac{x}{3}\cos\frac{x}{4}\cdots.$$

Prove that $F(x)\to 0$ as $x\to+\infty$. Prove in fact that $F(x)$ falls off exponentially.

Solution by N. J. Fine, University of Pennsylvania. For $x>0$, the number of positive integers k such that $\pi/3\leqq x/k\leqq 2\pi/3$ is at least $(3x/2\pi)-2$. Taking only these factors, we have

$$|F(x)| \leqq 4(\tfrac{1}{2})^{3x/2\pi} = O(e^{-cx}),$$

where $c=(\log 8)/2\pi$. This gives the desired result.

The constant can be improved considerably. Thus, by Vieta's product,

$$\frac{\sin x}{x} = \cos\frac{x}{2}\cos\frac{x}{4}\cos\frac{x}{8}\cdots$$

so

$$F\left(\frac{x}{2}\right) = \frac{\sin x}{x}\cdot\frac{\sin(x/3)}{x/3}\cdot\frac{\sin(x/5)}{x/5}\cdots.$$

Determine n so that $2n-1\leqq x\leqq 2n+1$. Then

$$\left|F\left(\frac{x}{2}\right)\right| \leqq \frac{1\cdot 3\cdot 5\cdots(2n-1)}{x^n} = \frac{(2n)!}{(2x)^n n!} \sim \frac{1}{\sqrt{2}}\left(\frac{2n}{ex}\right)^n,$$

by Stirling's formula. Thus $F(x/2)=O(e^{-n})=O(e^{-x/2})$, and $F(x)=O(e^{-x})$.

Also solved by Joseph Lehner, the Proposer, and one whose communication was unsigned. Late solution by Chih-yi Wang.

Compound Normal Distribution

4616 [1954, 718]. *Proposed by A. C. Cohen, Jr., University of Georgia*

Show that the compound normal distribution function

$$f(x) = \frac{1}{2\sigma\sqrt{2\pi}}\left[\exp\left\{-\tfrac{1}{2}\left(\frac{x-m_1}{\sigma}\right)^2\right\} + \exp\left\{-\frac{1}{2}\left(\frac{x-m_2}{\sigma}\right)^2\right\}\right]$$

is bimodal if $|m_1-m_2|>2\sigma$ and unimodal otherwise.

Solution by P. J. Burke, Bell Telephone Laboratories, New York City. Write $f(x)=f=f_1+f_2$ where

$$f_1 = \frac{1}{2\sigma\sqrt{2\pi}}\exp\left\{-\frac{(x-m_1)^2}{2\sigma^2}\right\}.$$

Take $m_2 \geqq m_1$. Except for the trivial case $m_1=m_2$, the extrema of f lie in the domain $m_1<x<m_2$ since only there does $\operatorname{sgn} f_1' = -\operatorname{sgn} f_2'$. Let $y=f'/f_1$ and let $k(*)$ be the number of distinct zeros of $*$. Then $k(y)=k(f')$. Setting $f'=0$, one has also $y=0$, and after transposing and taking logarithms in the latter equation, one has

$$z \equiv -\log(x-m_1) + \log(m_2-x) - \frac{[m_2^2 - m_1^2 - 2x(m_2-m_1)]}{2\sigma^2} = 0.$$

Since z has a continuous derivative in the domain defined above, $k(z)\leqq 1+k(z')$; and since z' is quadratic, $k(z)=k(f')\leqq 3$. Hence f has at most 3 extrema.

By direct substitution it is found that $x_c=\frac{1}{2}(m_1+m_2)$ is a root of $f'=0$, and by substitution in f'' it is found that $f(x_c)$ is a maximum for $|m_1-m_2|<2\sigma$ and a minimum for $|m_1-m_2|>2\sigma$. Since f is positive and continuous for $-\infty<x<+\infty$, symmetric in $x=x_c$ and asymptotic to the x-axis, therefore f must have two maxima when $f(x_c)$ is a minimum, and when $f(x_c)$ is a maximum, the unique mode must be at $x=x_c$, since f can have no more than three extrema. For $|m_1-m_2|=2\sigma$, $f(x_c)$ is a maximum since $f'''(x_c)=0$ and $f^{iv}(x_c)=-2/[\sigma^5(2\pi e)^{-1/2}]$.

Also solved by W. J. Blundon, P. G. Kirmser, M. S. Klamkin, S. Parameswaran, Nathan Shklov, Chih-yi Wang, and the Proposer.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to E. P. Vance, Oberlin College, Oberlin, Ohio.

Modern Trigonometry. By J. C. Brixey and R. V. Andree. New York, Henry Holt and Company, 1955. xii+154+Answers+Tables +Index. $3.25.

Plane Trigonometry. By C. R. Wylie, Jr. New York, McGraw-Hill Book Company, 1955. viii+281+Tables+Answers+Index. $4.00.

As far as it was possible to ascertain, the first of the above books is quite different from any of the many trigonometry textbooks now available. Many teachers and scientists were consulted in its preparation, and a lithoprinted edition was used in seventy-six college classes and an unstated number of high school classes. The book would thus seem to aim at both college and high school needs. Problem lists are ample, and answers and *hints* are given at the end of the book for a great number of problems, both odd and even-numbered. Figures also appear in the answers when these are called for in the problems. All figures are well-drawn, and those of the text are usually placed in the exceptionally wide, outer margins of the pages. Curves showing the beauty of geometric form decorate the ends of the chapters following chapter summaries and self-tests for the student. Chapters have no titles, and, while sections start with chapter and section numbers, these numbers are not shown on each page as is now a common custom.

Chapter 1 introduces functional notation and the distance formula as well as reviewing fractions, solution of equations, and Cartesian coordinates. The six trigonometric functions are defined for general angles in Chapter 2, where is found also a brief mention of radians. Chapter 3 deals with four-place logarithms commencing with the definitions and the laws of exponents. Scientific notation is employed to estimate numerical results and to explain significant figures and characteristics. The computer's rule for favoring the even digits is not mentioned, but logarithmic and exponential equations are treated. A folder of four pages with four-place tables of powers, logarithms, and natural functions is supplied. These tables are duplicated at the back of the book where there is also a four-place table of logarithmic trigonometric functions. The greatest departures from conventional treatment occur in Chapters 4 and 9. The former treats the solution of triangles. The student is told how to use his tables. The cosine and sine laws are derived, and the tangent law is stated without proof. The solution of right triangles is not mentioned, nor is it illustrated by solved examples. Nevertheless, problems on this topic are given. The usual four cases of oblique triangles are missing. No illustrative examples are solved with logarithms, and the data of the problems seem to lend themselves to arithmetic. A teacher might, therefore, omit the chapter on logarithms. The same could be said of Chapter 9 which seems to be recreational, giving very brief discussions of several topics such as matrices, angle trisection, the four color problem, and finite geometry.

Chapter 5 treats the graphs of all six trigonometric functions and emphasizes the fact that their arguments are real variables which need not represent angles. The analytic part of trigonometry is usually the most difficult for the student. Placing this near the end in Chapter 6 should have its advantages. The distance formula is used to prove the composite angle formulas, a method recently characterized by a reviewer in this MONTHLY as the first completely satisfactory one in elementary textbooks. Arcsine, arccosine, and arctangent are defined without reference to their graphs; other inverse trigonometric functions are mentioned but not defined. Chapters 7 and 8 contain topics that might be included in a longer-than- usual course. The former deals with polar coordinates, and the latter treats complex numbers and De Moivre's theorem.

When two books are being discussed in the same review, there is an obvious temptation to make comparisons. When these occur, they are not to be construed as derogatory to either book. Our first book has about the average number of pages. Our second exceeds it by one hundred and seventy pages. Hence the latter has more material, and its explanations are, in general, more detailed. Chapter and section numbers appear on each page. Exercises and illustrative examples are printed in smaller type than the text, and answer to the odd-numbered exercises appear at the end of the book. One notices the use of numerous footnotes, and the mention of many connections between trigonometry and plane geometry.

Chapter 2 has a clear discussion of the logic used in proving identities. Chapter 3 with thirty-four pages is entitled *The Solution of Right Triangles*. It uses only arithmetic and has many applied problems. Chapter 4, *The Graphs of the Trigonometric Functions*, has a good motivating discussion of the applications of $y=A \sin (bx-c)$ and $y=A \cos (bx-c)$. A briefer treatment of the graphs of the other trigonometric functions is followed by a section of four pages on polar coordinates. A derivation of the distance formula commences Chapter 5 in which this formula is used to obtain the formulas for functions of composite angles. Chapter 6 on inverse functions begins with a very helpful exposition of inverse functions in general and their graphs. The principal values of the inverse trigonometric functions are explained from their graphs and are indicated by the usual initial capitals. At the beginning of Chapter 7, *Trigonometric Equations*, the author states, "We shall give all solutions wherever possible and then indicate which one of the set is the principal value." The discussion is more complete than usual, and it includes equations with inverse functions and those involving both algebraic and trigonometric functions. Chapter 8, *Logarithms*, starts with a very brief review of exponents followed by the general definition of a logarithm and the laws of operation with logarithms. Logarithmic solution of right triangles is included in the chapter as are change of base, natural logarithms, logarithmic and exponential functions. The solution of equations involving algebraic and exponential functions is illustrated by examples. All logarithmic tables are five-place. Chapter 9 is *Solution of Oblique Triangles*. The distance formula is used to derive the law of cosines which is then employed to solve

cases III and IV arithmetically. The familiar four cases (although not numbered) are solved by logarithms in the usual manner. Mollweide's formula is given as "probably the best independent check." Chapters 10, 11, and 12 contain material for which there is no time in the usual freshman course, but they are well and interestingly written. Their titles are, respectively, *Complex Numbers and De Moivre's Theorem*, *Trigonometric Series*, and *Hyperbolic Functions*.

R. F. GRAESSER
University of Arizona

The Elements of Probability Theory and Some of Its Applications. By Harald Cramér, New York, J. Wiley and Sons, 1955. 281 pages, $7.00.

This book is a revised and extended translation of the author's earlier Swedish text. It is an elementary treatise for beginners who have a modest mathematical background, namely a familiarity with analytic geometry, calculus and some algebra including determinants. Therefore, the book does not aim at a complete and rigorous mathematical development but refers often for proof to other books, primarily to the author's *Mathematical Methods of Statistics* (Princeton University Press, 1946).

The book is divided into three parts. Part I (Chapters 1–4, 44 pages) deals with the foundations. Chapter 1 gives a brief historical survey. In Chapter 2 the author discusses phenomena which exhibit statistical regularity. The aim of the book is to construct a mathematical model for the description and interpretation of such phenomena. Clearly, the level for which the book is intended precludes a rigorous axiomatic treatment of probability theory. Nevertheless the book is written in the spirit of the foundations of Kolmogorov. A student who starts his study of probability theory with this book should gain the proper appreciation for the modern viewpoint and should be able to read more advanced treatises with ease since a motivation for the modern approach is implicitly provided by the present book. In Chapter 3 the author proceeds to discuss some basic properties and concepts of the mathematical theory of probability (the addition and multiplication rules, conditional probabilities, independent events, Bayes theorem) and in Chapter 4 applies these to the study of repeated observations, drawings with and without replacements and simple problems on games of chance.

Part II (Chapters 5–10, 95 pages) has the title *Random Variables and Probability Distributions*. In Chapter 5 random variables are presented as results of random experiments and the concept of a distribution function is introduced. The two most important types of distribution functions, namely, discrete and continuous distributions are studied in greater detail. Some basic ideas connected with multivariate distributions are considered. The usual measures of location, dispersion and skewness, as well as mean values and moments, are defined and Tchebycheff's inequality is proven. Here, as well as in other instances, the author states the theorem in full generality with a proof which applies either to the continuous or the discrete case. The second proof is omitted and is left as an

exercise to the reader. The addition of independent random variables is discussed and generating functions are defined. The problem of the existence of a generating function is stated and it is mentioned that this difficulty could be avoided by introducing characteristic functions. However, no further use is made of characteristic functions since the author wishes to avoid complex variables. The chapter concludes with a discussion of some of the important properties of generating functions. Chapter 6 deals with the binomial distribution and related topics. Bernoulli's law of large numbers and de Moivre's form of the central limit theorem are discussed. The Poisson distribution is introduced as a limiting distribution for the binomial distribution. The normal distribution and a more general statement of the central limit theorem are treated in Chapter 7. Chapter 8 deals with continuous distributions which are related to the normal. Sections of this chapter are devoted to the chi-square distribution, the t-distribution and the Pearson system. The last two chapters of Part II contain essential material concerning bivariate and multivariate distributions.

Part III (Chapters 11–16, 108 pages) has the title *Applications*. In the introductory Chapter 11 the author clarifies the aims and limitations of statistical tests for investigating the agreement between some suggested theoretical model and actual observations. Chapter 12 deals with the descriptive treatment of statistical material. The empirical distribution of the sample is defined and the distinction between the characteristics of the population and the sample characteristics is emphasized. Chapter 13 introduces the idea of sampling distributions. Moments of the sampling distribution of certain characteristics, for example the mean and the variance, are used to estimate the corresponding population parameters. The asymptotic normality of the distribution of a wide class of sample characteristics is mentioned with an appropriate reference to the detailed treatment in the author's *Mathematical Methods of Statistics*. Certain order statistics are given as examples of exceptions. Thus the reader is warned that it is wrong to assume that the distributions of *all* sampling characteristics are asymptotically normal. This is but one example of the author's careful exposition which avoids the evils of "cook-booking" even though he is forced to omit certain proofs which could not be understood by a reader with the assumed mathematical background. This chapter concludes with a derivation of the exact distribution of the mean and of the variance of a sample drawn from a normal population. The subject of Chapter 14 is statistical inference. A very concise and brief discussion of the problem of estimation and of certain properties (unbiasedness, consistency, efficiency) of estimates is given. The maximum likelihood method as well as the method of moments is presented. The concept of confidence intervals is introduced and discussed in some detail for samples from normal populations. This is supplemented by the consideration of cases where it is known that the estimates involved are only asymptotically normal. The chapter closes with some remarks on the Neyman-Pearson theory of testing hypothesis. Fleeting remarks are made concerning sequential analysis, the concept of risk and A. Wald's theory of statistical decision functions. The purpose

of these remarks is to indicate the existence of further developments and to give the reader stimulation and references for the continuation of his studies. Chapter 15 treats the chi-square test, proofs are mostly omitted but interesting illustrative examples are given. Chapter 16 deals with a variety of topics, such as the theory of errors and regression problems (17 pages), the analysis of variance (10 pages), sampling and quality control (9 pages). This chapter gives an idea of the wide scope of applications of statistical methods. There is a brief appendix with some formulae concerning the gamma function, a table of references, answers to the problems and some statistical tables (normal, chi-square, t- and F-distributions). Problems are inserted at the end of almost every chapter. A large number of interesting illustrative examples are worked out in detail throughout the text. Among these, as well as among the problems, one finds also some interesting applications to mortality tables and to insurance problems.

This reviewer regrets that the third part is restricted to the discussion of parametric methods. He feels that the inclusion of some material on nonparametric tests would have been desirable and feasible. He hopes that this will be added to future editions of this excellent book.

The author intended to write a text book for an introductory course which could be presented to students with a modest background and a variety of motives for studying probability theory and its applications. This very difficult aim has been accomplished with great success and this reviewer feels that this text will help to promote better teaching of elementary probability and statistics.

EUGENE LUKACS
Office of Naval Research, Washington, D. C.

An Introduction to Stochastic Processes. By M. S. Bartlett. New York, Cambridge University Press, 1955. 14+312 pages. $6.50.

The question which recurred often to this reviewer while going through Bartlett's book is, "An introduction for whom?" According to the preface, the volume is addressed to applied mathematicians and statisticians, emphasis being on "methods and applications." The author's plan necessitates omission of much rigor and detail. A treatment of this nature could serve as a useful introduction to the subject for many readers (applied mathematicians and statisticians, as well as others) who want to obtain some idea of the applicability of the theory without spending time on a full mathematical treatment à la Doob; however, it seems doubtful that the present volume will prove to be such a useful introduction for many of these readers.

A main criticism of the book is that omission of some rigor and detail should not necessitate hazy (or complete omission of) definitions of words used technically in the text, and which make whole sections difficult (if not impossible) to decipher. For example, most readers will be confused by the remarkable phrasing in the definition of a "nearest neighbour system" on page 35. In the sections on statistical inference which constitute the last part of the book, the uninitiated

will wonder at many undefined phrases like (page 265) "simple consistent (though not fully efficient) equations of estimation," as well as why one should use certain proposed tests of significance (the notion of the power of a test is never mentioned).

Other aspects of the author's style are also disturbing or fatiguing. One is continually referred to an unpublished work by Moyal for material which can be found in published but unmentioned sources. The book has several disturbing errors; for example, the sufficient condition for the central limit theorem stated on page 9 is incorrect. The mathematical level and choice of what is or is not to be abbreviated sometimes seem uneven, but some of this may be a matter of taste.

On the positive side, there is a great variety of applications in the book. The topics considered range from epidemic models and harmonic analysis to sequential analysis and queueing theory, and include many others. Any usefulness of the book seems to this reviewer to be found in this variety of topics. Their treatment suffers from the defects noted above, and the direction of these topics may not prove useful to all readers (for example, readers concerned with applications to engineering and physics will probably find the book of Blanc-Lapierre and Fortet more useful); but Bartlett's book may prove a useful (if incomplete) source of references and summary of results in some areas. Thus, for example, a reader interested in population growth can find in this book a list of various models and results, a sketch of several methods of obtaining such results, and a list of some papers which contain more detailed presentations (that reader will often find the references more readable than the book). However, for an "introduction" to the subject which will give him an understanding of the basic ideas, a reader interested primarily in applications without too much heavy mathematics would still do better to read (*e.g.*) appropriate portions of Feller's book in place of corresponding sections of Bartlett's book. There is no substitute for a clear, precise presentation of the elementary concepts of a subject like this one for a reader who wants to understand fully the applications.

J. KIEFER
Cornell University

NEW BOOKS RECEIVED

Vector Analysis (International Series in Pure and Applied Mathematics). By H. E. Newell, Jr. New York, McGraw-Hill Book Company, 1955. 11+216 pages. $5.50.

The Geometry of Geodesics. By Herbert Busemann. New York, Academic Press, Inc., Publishers, 1955. x+422 pages. $9.00.

A Manual of Problems in Statistics, Revised Edition. By Scott Dayton. New York, Henry Holt and Company, 1955. v+137 pages. $1.95.

Memorial des Sciences Mathematiques. Vol. 128. La sommation des series divergentes. By Marc Zamansky, 1954. Paris, Gauthier-Villars, 79 pages.

Collection de Logique Mathematique, Series A., No. 7, Deux Esquisses de Logique. By J. B. Rosser. Paris, Gauthier-Villars, 1955. 70 pages. 900 fr.

Colloque sur L'Analyse Statistique. (Held at Brussels Dec. 15, 16, 17, 1954). Paris, Georges Thone Liege, Masson et Cie, 1955. 186 pages. 1.900 fr. francs.

Cours de Geometrie Infinitesimale, 5th vol. Geometrie Infinitesimale 2nd Part, Theorie des Surfaces. By Gaston Julia. Paris, Gauthier-Villars, 1955, 145 pages.

Cours de Geometrie Infinitesimale. 4th Vol. Cinematique et Geometrie Cinematique, 2nd part, Etude Approfondie du Mouvement d'un Corps Solide. By Gaston Julia. Paris, Gauthier-Villars, 1955. 88 pages.

Cours de Geometrie Infinitesimale. 3rd Vol. Geometrie Infinitesimale. 1st part. Methodes Generales. Theorie des Courbes. By Gaston Julia. Paris, Gauthier-Villars, 1955. 220 pages.

Theoreme sur les Surfaces D'onde en Optique Geometrique, avec une note sur le Miroir Integral. By Rene Damien. Paris, Gauthier-Villars, 1955. 34 pages. $2.66.

College Algebra and Trigonometry, A Basic Integrated Course. Second Edition. By F. H. Miller. New York, John Wiley and Sons, Inc., 1955. xiv+342 pages. $4.50.

International Encyclopedia of Unified Science, Vol. 1, part 1, and part 2. Edited by Otto Neurath, Rudolph Carnap and Charles Morris. Chicago, Illinois, The University of Chicago Press, 1955. *Part 1*, ix+339 pages; *Part 2*, 342–760 pages. $6.00 each. (If purchased together, the set costs $11.00.)

Astrophysical Quantities. By C. W. Allen. London, England, University of London, The Athlone Press, 1955. xi+263 pages. $10.00. (Distributed in America by John de Graff, Inc., 64 W. 23rd St., New York 10, New York.)

Elementary Statistics for Students of Social Science and Business. By R. C. Sprowls. New York, McGraw-Hill Book Company, 1955. xiii+392 pages. $5.50.

Basic Mathematics for Science and Engineering. By P. G. Andres, H. J. Miser and H. Reingold. New York, John Wiley and Sons, Inc., 1955. vii+846 pages. $6.75.

Shop Mathematics. By C. E. Stout. New York, John Wiley and Sons, Inc., 1955. xi+282 pages. $3.70.

Reflections of a Physicist. Second Edition. By P. W. Bridgman. New York, Philosophical Library, 1955. xiv+576 pages. $6.00.

A Manual of Problems in Statistics, Revised Edition. By Scott Dayton. New York, Henry Holt and Company, 1955. 137 pages.

Analytic Geometry. Third Edition. By Sisam and Atchison. New York, Henry Holt and Company, 1955. xxiv+292 pages. $3.75.

Principles of Mathematics. By C. B. Allendoerfer and C. O. Oakley. New York, McGraw-Hill Book Company, 1955. xv+448 pages. $5.00.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Bradley University makes the following announcements: Mr. K. E. Lake of Washburn Municipal University has been appointed to an assistant professorship; Assistant Professor Joseph R. Brown has been promoted to an associate professorship.

Lehigh University reports the following: Mr. Gerhard Rayna, previously a programmer at the Forrestal Research Center, Princeton, New Jersey, Dr. Irving Weiss, formerly a statistician at Sylvania Electric Products Company, and Mr. F. O. Wyse, previously an assistant in instruction at Princeton University, have been appointed to instructorships; Assistant Professor Chuan-Chin Hsiung has been promoted to an associate professorship.

At McGill University: Assistant Professor J. E. L. Peck of the University of New Brunswick has been appointed to an associate professorship; Mr. G. D. Findlay, Mr. J. E. LeBel, Mr. J. Michael, Dr. T. V. Narayana, and Mr. L. A. Smith have been appointed Lecturers.

Texas Christian University announces: Dr. Mabel G. Reavis has been appointed to an assistant professorship; Associate Professor L. A. Colquitt has been promoted to a professorship.

Tulane University reports: Professor B. J. Pettis has been appointed Chairman of the Department of Mathematics of the University for the year 1955–1956; Dr. F. B. Wright, Jr., has been promoted to an assistant professorship; Associate Professor A. H. Clifford of Johns Hopkins University has been appointed Head of the Department of Mathematics of Newcomb College; Mr. F. L. Cleaver, formerly a teaching assistant at the University of Miami, Mr. C. P. Gadsden, previously a teaching assistant at the University, and Miss Eugenia I. Trapp, recently a graduate teaching assistant at the University, have been appointed to instructorships; Dr. Morikuni Goto, formerly a research associate at the University of Illinois, has been appointed to an assistant professorship; Assistant Professor Paul Conrad has been granted a Fulbright award for lecturing at the University of Colombo, Ceylon, for the academic year 1956–1957; Mr. A. B. Simon, a graduate fellow, has been awarded a National Science Foundation Fellowship and is continuing his graduate work at the University.

University of California at Los Angeles announces the following: Dr. P. C. Curtis, Jr., has been appointed to an instructorship; Dr. C. B. Tompkins, director of numerical analysis research, has been appointed to a professorship; Assistant Professors L. J. Paige and E. G. Straus have been promoted to associate

professorships; Associate Professor F. A. Valentine has been promoted to a professorship.

At the University of Illinois: Dr. J. R. Buchi of the University of Michigan, Dr. D. L. Burkholder, previously a graduate research assistant at the University of North Carolina, Dr. N. T. Hamilton of Cornell University, and Mr. J. A. Zilber of Johns Hopkins University have been appointed to assistant professorships; Dr. Klaus Krickeberg, previously a lecturer at Universitat Wurzburg, has been appointed Research Associate; Dr. W. M. Zaring, formerly a graduate assistant at the University of Kentucky, has been appointed to an instructorship; Assistant Professors C. R. Blyth and W. A. Ferguson have been promoted to associate professorships; Associate Professor C. W. Mendel has been promoted to a professorship; Dr. Jewell E. Schubert has been promoted to an assistant professorship; Dr. Michio Suzuki, recently a research associate, has been promoted to an assistant professorship.

University of Miami reports the following: Dr. E. F. Low, Jr., formerly an aeronautical research scientist for the National Advisory Committee for Aeronautics, Langley Field, Virginia, and Dr. R. V. Mendenhall, previously a senior mathematician at Vitro Laboratories, Eglin Air Force Base, Florida, have been appointed to assistant professorships; Mr. Frederick Borges has been appointed to an instructorship.

University of Notre Dame announces: Dr. Irving Glicksburg, previously a research associate for the Rand Corporation, and Dr. William Huebsch, previously a National Science Foundation postdoctoral fellow at the Institute for Advanced Study, have been appointed to assistant professorships; Dr. R. C. Taliaferro has been appointed to an associate professorship; Mr. Frederick Bagemihl, formerly a research fellow at the Institute for Advanced Study, has been appointed Visiting Lecturer.

At the University of Pittsburgh: Mr. Samuel Hanna and Mr. Earl McKinney have been appointed to instructorships.

University of Rochester reports: Mr. J. P. Line of Oberlin College and Mr. Edward Wong, formerly a graduate teaching assistant at the University, have been appointed to instructorships; Assistant Professor Walter Rudin has been promoted to an associate professorship; Mr. R. A. Raimi has been promoted to an assistant professorship and is on leave of absence during 1955–1956.

Assistant Professor W. F. Atchison of the University of Illinois has a position at the Rich Electronic Computer Center, Georgia Institute of Technology.

Dr. V. N. Behrns of Consolidated Vultee Aircraft Corporation, Fort Worth, Texas, has a position as an operations analyst with Technical Operations, Arlington, Massachusetts.

Assistant Professor R. L. Beinert of Hobart and William Smith Colleges has been promoted to an associate professorship.

Dr. Stoughton Bell of the University of California has accepted a position as a staff member with the Sandia Corporation, Albuquerque, New Mexico.

Assistant Professor W. G. Brady of Washington and Jefferson College has been appointed Acting Head of the Department of Mathematics.

Mr. R. E. Cornish of Seton Hall University has accepted a position as a programmer with International Business Machines Corporation, Kingston, New York.

Mr. L. J. Derr, formerly of Tulane University, has a position as a mathematical analyst with the Shell Oil Company, New Orleans, Louisiana.

Dr. William H. Durfee of the Operations Research Office has been appointed to an associate professorship at Mount Holyoke College.

Associate Professor L. A. Dye of The Citadel has been promoted to a professorship.

Dr. R. E. Graves of the Goodyear Aircraft Corporation is now a member of the technical staff of the Ramo-Wooldridge Corporation, Los Angeles, California.

Dr. J. S. Griffin, Jr., of Tulane University has been appointed to an instructorship at the University of Michigan.

Dr. J. F. Heyda of the Naval Ordnance Plant has accepted a position as a mathematician with the Allison Division of General Motors, Indianapolis, Indiana.

Dr. M. A. Hyman of the Westinghouse Atomic Power Division, Pittsburgh, Pennsylvania, has accepted a position as a research analyst with the Eckert-Mauchly Division of the Sperry Rand Corporation, Philadelphia, Pennsylvania.

Dr. J. R. Isbell of Tulane University is now in military service.

Mr. A. A. Karwath of the State University of Iowa has been appointed to an assistant professorship at the United States Naval Academy.

Mr. Raymond Kassler of Boland and Boyce, Belleville, New Jersey, has a position as a development engineer with the Atlantic Design Company, Newark, New Jersey.

Associate Professor Dora E. Kearney of Upper Iowa University has been appointed to an assistant professorship at Iowa Wesleyan College.

Assistant Professor George Klein of Mount Holyoke College has been appointed to an assistant professorship at Tufts College.

Dr. A. B. Lehman has been appointed a staff member in the Acoustics Laboratory, Massachusetts Institute of Technology.

Assistant Professor W. J. LeVeque of the University of Michigan has been promoted to an associate professorship.

Associate Professor R. J. Levit of the University of Georgia has accepted a position in the Applied Science Division of International Business Machines Corporation, San Francisco, California.

Mr. E. S. Lowry, formerly a student at the University of Toronto, has a position as a programmer at Computing Devices of Canada, Ottawa, Ontario, Canada.

Mr. G. E. Mahoney has a position as a mathematician with the Computer Control Company, Point Mugu, California.

Dr. Imanuel Marx of the University of Michigan has been promoted to an assistant professorship.

Dr. Elliott Mendelson has been appointed to an instructorship at the College of the University of Chicago.

Assistant Professor M. W. Milligan of Adams State College has been promoted to an associate professorship.

Assistant Professor V. H. Morrill of Hardin-Simmons University has been appointed to an acting assistant professorship at Tarleton State College.

Mr. C. R. Morris of the University of Texas has accepted a position as an associate engineer with the Lockheed Aircraft Corporation, Burbank, California.

Mr. G. M. Muller, formerly a physicist at General Electric Company, Richland, Washington, has accepted an appointment as a mathematician at Stanford Research Institute, Menlo Park, California.

Dr. D. V. Newton of the University of Washington is now an applied science representative with the International Business Machines Corporation.

Dr. Albert Nijenhuis, formerly a member of the Institute for Advanced Study, has been appointed Research Associate and Instructor at the University of Chicago.

Mr. S. J. Scott, formerly a student at the University of North Carolina, has a position as a mathematician with the Vitro Corporation of America, Eglin Air Force Base, Florida.

Mr. W. B. Simmons, Jr., recently a student at Knox College, is employed as an associate engineer in the Mathematical Analysis Department, Lockheed Aircraft Corporation, Burbank, California.

Dr. N. B. Smith of San Diego State College has been promoted to an assistant professorship.

Professor Irving Sussman of the University of San Francisco has been appointed to a professorship at California State Polytechnic College.

Mr. O. E. Taulbee, previously an assistant at Michigan State University, has a position as a mathematician in the Systems Analysis Group, Remington Rand, Engineering Research Associates Division, St. Paul, Minnesota.

Assistant Professor L. C. Teng of the University of Wichita has accepted a position as an associate physicist at the Argonne National Laboratory, Lemont, Illinois.

Professor C. J. Thorne of the University of Utah has accepted a position as a research scientist with the Naval Ordnance Test Station, China Lake, California.

Associate Professor R. M. Thrall of the University of Michigan has been promoted to a professorship.

Associate Professor Emeritus Edith I. Atkin of Illinois State Normal University died on August 27, 1955. She was a member of the Association for twenty-nine years.

Professor C. A. Messick, formerly of Park College, died on October 10, 1955. He was a member of the Association for thirty years.

Professor S. B. Myers of the University of Michigan died on October 8. 1955.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 77 persons have been elected to membership by the Board of Governors on applications duly certified.

LINDA ALLEGRI, A.M. (Hunter) Mathematics Teacher, Hunter College High School.

F. W. ANDERSON, Ph.D. (S.U. of Iowa) Instr., University of Nebraska.

CAPT. R. A. AWTREY, U.S.N. (ret.), M.A. (Florida) Instr., University of Florida.

F. C. BARNETT, M.S. (New Mexico) Instr., Rose Polytechnic Institute.

L. C. BARRETT, M.S. (Utah) Instr., University of Utah.

ALEXANDER BECK, B.S. in Ed. (Temple) Teacher, Olney High School, Philadelphia, Pa.

J. M. BEHR, Student, University of California at Los Angeles; Technical Writer, Librascope, Glendale, Calif.

I. O. BENTSEN, B.A. (Hobart) Grad. Student, University of Rochester.

R. E. BRENNAN, B.S., B.A. (Amherst) Representative, Addison-Wesley Publishing Co., Cambridge, Mass.

LOUIS BRICKMAN, Student, Temple University.

F. W. BRINKLEY, JR., B.A. (Denver) Grad. Student, University of Denver.

ARMAND BRUMER, Student, Brandeis University.

H. L. COOK, Ph.D. (Peabody-Vanderbilt) Head, Department of Mathematics, West Texas State College.

B. G. COOPER, B.A. (Stephen F. Austin S.C.) Grad. Asst., Kansas State College.

J. N. CRAWFORD, M.S. (East Texas S.T.C.) Head, Department of Mathematics, LeTourneau Technical Institute.

C. W. CURTIS, Ph.D. (Yale) Asst. Professor, University of Wisconsin.

A. G. DAVIS, M.A. (Harvard) Asst. Professor, Clarkson College of Technology.

R. V. DIMEO, A.B. (Rutgers) Atlantic City, New Jersey.

R. E. DOWD, M.A. (N.Y.U.) Garnac Grain Co., New York, N.Y.

CHURCHILL EISENHART, Ph.D. (London) Chief, Statistical Engineering Lab., National Bureau of Standards, Washington, D. C.

J. D. ERDWINN, B.A. (Rice) Grad. Asst., Rice Institute.

W. M. FAUCETT, Ph.D. (Tulane) Senior Aerophysics Engr., Consolidated-Vultee Aircraft Corp., Fort Worth, Texas.

CONSTANCE M. FOLEY, M.A. (New Hampshire) Instr., West Virginia University.

I. C. FRANCIS, B.S. (Puget Sound) Seattle, Washington.

D. A. FREEDMAN, Student, McGill University.

D. G. FULTON, Ph.D. (Michigan) Professor, Tufts University.

DAVID GALE, Ph.D. (Princeton) Asst. Professor, Brown University.

J. J. GOODE, Student, Georgia Institute of Technology.

MRS. DOROTHY H. HELMICK, M.S. (Drake) Instr., Drake University.

J. R. HENDRICKS, B.A. (British Columbia) Meteorologist, Aviation Forecast Office, Winnipeg, Manitoba.

A. H. HERRINGTON, A.M. (Duke) Superintendent, Quitman Public Schools, Ga.

F. W. HOAGLAND, Chicago, Illinois.

MRS. RUTH M. HOFSTRA, M.A. (Syracuse) Fellow, University of Tennessee.

J. W. JOY, B.A. (Washington S.C.) Research Asst., Scripps Institution of Oceanography, La Jolla, Calif.

R. C. JURGENSEN, M.S. (Iowa) Chairman, Department of Mathematics, Culver Military Academy, Ind.

T. F. KIMES, B.S. (Ursinus) Teaching Asst., University of Texas.

GEORGE KOLETTIS, JR., M.S. (Chicago) Instr., Northwestern University.

H. E. KREHBIEL, M.S. (Michigan S.U.) Mathematics Teacher, Niles High School, Mich.

J. B. LACKEY, B.S. (Eastern Kentucky S.C.) Part-time Instr., University of Kentucky.

C. L. LAWSON, B.S. (California) Optometry Officer, U. S. Army Hospital, Fort Jackson, S. C.

F. R. LIDDIL, M.A. (Chicago) Instr., Illinois Institute of Technology.

MRS. SYLVIA C. P. LU, B.S. (Taiwan T.C.) Instr., Taipei College of Technology, Taiwan, China.

T. A. MACKEY, Student, Georgetown University.

A. G. MAZZA, B.S. (London) Senior Mathematics Master, Scottish Schools, Alexandria, Egypt.

J. E. MCFARLAND, M.S. (Oregon S.C.) Grad. Asst., Oregon State College.

ELLIOTT MENDELSON, Ph.D. (Cornell) Instr., University of Chicago.

D. W. MILLER, M.S. (Wisconsin) Instr., University of Nebraska.

LOIS B. MITCHELL, A.B. (Vassar) Mathematician, I.B.M. Corp., New York, N. Y.

MRS. SHU-TEH C. MOY, Ph.D. (Michigan) Instr., Wayne University.

N. W. NAUGLE, B.A. (A.&M.C. of Texas) Teaching Fellow, Agricultural and Mechanical College of Texas.

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R. R. PEREZ, Student, University of Denver.

MARVIN ROSENBLUM, Ph.D. (California) Acting Asst. Professor, University of Virginia.

V. J. RYBA, B.S. (St. Bonaventure) Grad. Student, University of Buffalo.

R. T. SANDBERG, B.A. (Alfred) Teaching Fellow, University of Buffalo.

E. C. SCHLESINGER, Ph.D. (Harvard) Instr., Yale University.

J. E. SCROGGS, M.A. (Houston) Asst., Rice Institute.

S. P. SHEN, A.B. (Assumption) Scholar in Physics, Clark University.

VIRGINIA SHUFORD, M.A. (Appalachian S.T.C.) Professor, Reinhardt College.

H. T. SLABY, Ph.D. (Wisconsin) Instr., Wayne University.

J. R. SMART, Student, San Jose State College.

BERNARD SMILOWITZ, Pittsburgh, Pennsylvania.

D. A. STORVICK, Ph.D. (Michigan) Instr., Iowa State College.

D. J. STRUIK, Ph.D. (Leiden) Professor, Massachusetts Institute of Technology.

PATRICK SUPPES, Ph.D. (Columbia) Asso. Professor, Stanford University.

ROBERT TATES, B.S. (Rochester) Grad. Student, Syracuse University.

SCOTT TAYLOR, Student, University of California, Berkeley.

W. F. TRENCH, A.M. (Pennsylvania) Asst. Instr., Electrical Engineering, University of Pennsylvania.

T. E. WADDILL, B.S. in Ed. (S.T.C., Kirksville, Mo.) Instr., Chillicothe High School, Mo.

R. C. WARNER, B.A. (Toronto) Grad. Asst., University of Rochester.

J. H. WELLS, M.S. (Texas Tech. C.) Teaching Asst., University of Texas.

D. H. WENTWORTH, M.S. (North Dakota) Instr., University of North Dakota.

NANCY E. WILHELM, B.S. (Oklahoma A.&M.C.) Engg. Asst., General Electric Co., Cincinnati, Ohio.

REV. J. J. WINDOLPH, M.S. (Notre Dame) Instr., Quincy College.

MARY E. B. WIRTZ, M.S. (Louisiana S.U.) Instr., McNeese State College.

R. L. YORK, Student, Siena College.

THE OCTOBER MEETING OF THE MINNESOTA SECTION

The Fall meeting of the Minnesota Section of the Mathematical Association of America was held at South Dakota State College in Brookings, South Dakota, on October 15, 1955. Professor H. B. MacDougal presided at the morning session, and Professor E. J. Camp, Chairman of the Section, presided at the afternoon session.

There were 35 persons in attendance, including 22 members of the Association.

By invitation of the Executive Committee, Professor P. C. Rosenbloom of the University of Minnesota delivered an address at the morning session entitled "On the Fundamental Theorem of Algebra." Abstract of this address follows:

If the complex numbers are defined in terms of Cauchy sequences of complex rationals, then the statement that the complex number field is algebraically closed takes the form: If $P_n(z) = \sum_{k=0}^{N} a_{kn}z^k$ is a sequence of polynomials with complex rational coefficients such that $a_{Nn}=1$ for all n and $\{a_{kn}\}$ is a Cauchy sequence for $0 \leq k \leq N$, then there is a Cauchy sequence $\{z_n\}$ of complex rationals such that $P_n(z_n) \to 0$. It is easy to prove this by elementary algebraic methods if one has already shown that for any non-constant polynomial $P(z)$ with complex rational coefficients there is a complex rational z such that $|P(z)| < \epsilon$. The author has given an elementary algebraic proof of this using methods suggested by analytic function theory. It would be desirable to find a direct elementary proof of this lemma which is really a proposition in elementary number theory.

The following short papers were presented:

1. *An application of curve fitting to a problem in acoustics*, by Professor H. L. Black, North Dakota Agricultural College, introduced by Professor A. G. Hill.

For some types of transducers the pattern function, given theoretically in terms of Bessel functions, is commonly approximated trigonometrically. In a particular case it has been found convenient to assume a form,

$$b(0, \phi) = \cos^k n\phi.$$

By proper choice of the parameters, a good fit to the pattern is obtained for a considerable angular distance from the acoustic axis. In reverberation problems use of $b(0, \phi)$ leads to definite integrals of the Wallis type. Closely related problems give rise to somewhat more general definite integrals which are evaluated in terms of gamma functions.

2. *Two illustrations of the application of game theory*, by Professor R. P. Winter, College of St. Thomas.

This paper presents a brief report on the content of *The Compleat Strategyst* by J. D. Williams. Two problems with solutions as given in this neat primer of game theory are discussed.

3. *An application of the Laplace transform*, by Dean H. M. Crothers, South Dakota State College, introduced by Professor H. B. MacDougal.

In this paper the Laplace transform is applied to obtain a solution of a problem concerning an impact power wrench.

4. *On Hölder's inequality*, by Professor R. J. Dowling, College of St. Thomas.

The following inequality is used in text books to prove Hölder's inequality:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

where a, b, p and q are positive numbers and $1/p+1/q=1$. Geometric interpretations of this inequality were presented.

5. *Some remarks on divisibility*, by Professor F. L. Wolf, Carleton College.

The fact that a rational number can be expressed as a repeating decimal may be used to show that every positive integer divides some number that is a string of 9's followed by a string of 0's. Application of Fermat's little theorem gives easily that every prime greater than 5 divides some number that is a string of 1's. In listing the primes different from 2 and 5 by finding the divisors of 11, 111, 1111, . . . , does each number yield at least one prime not already listed?

6. *Report on the Madison Conference*, by Professor K. W. Wegner, Carleton College.

This conference was different from others sponsored by the National Science Foundation in that its emphasis was on teacher education. Only about a quarter of those attending were college teachers. Over a third of the scheduled meetings was devoted to studies in curriculum and instruction, mainly at the secondary school level.

This report included a presentation of a new proof of the theorem which states that every real positive number has a real kth root; this proof was given by F. Kiokomeister in one of the lecture courses at the Madison Conference.

F. C. SMITH, *Secretary*

THE OCTOBER MEETING OF THE OKLAHOMA SECTION

The Fall meeting of the Oklahoma Section of the Mathematical Association of America was held at Oklahoma City University, Oklahoma City, Oklahoma, on October 28, 1955. Professor R. R. Murphy, Chairman of the Section, presided.

There were 124 in attendance, including 84 members of the Association. This meeting is the annual meeting at which papers of special interest to high school and freshman and sophomore college teachers are presented. Research type papers are presented at a later meeting.

The following officers were elected for one-year terms: Chairman, Professor Truman Wester, Central State College; Vice-Chairman, Professor Paul Sanders, Southeastern State College; Secretary-Treasurer, Professor R. V. Andree, University of Oklahoma. The business meeting included also a discussion of proceedings at the meeting of section officers at Ann Arbor, Michigan. The group voted to continue meeting in conjunction with the Oklahoma Educational Association in the Fall and to hold the Spring 1956 meeting in Tulsa, Oklahoma. A motion to investigate the possibility of holding the second meeting during the 1956–57 academic year in conjunction with the Oklahoma Academy of Science, if possible, passed.

The following papers were presented:

1. *Classroom notes in elementary integral calculus*, by Professors Simon Green and M. Schwartz, University of Tulsa.

The attempt was made to fill part of the gap arising in most elementary books introducing students to the definite integral. Firstly, the books select regular intervals and secondly they demonstrate the simplest type of problem as a limit of the sum of a set of rectangles bounded from above and below. They leave the more difficult problems as exercises to be performed by the students who usually have not the necessary experience to cope with them. Examples bringing out clearly the independence of the mode of subdivision and solutions of problems of a more difficult nature were demonstrated.

2. *Publications of books and other materials needed in mathematics classes on the high school level*, by Miss Vivian Spurgeon, Nowata High School, Oklahoma.

Publications and other materials needed by the mathematics classes on the high school level were discussed. The materials needed are (1) supplementary materials for class work and (2) enrichment and general background materials. Teachers of high school mathematics should help prepare publications.

3. *A Carnegie Foundation project in modern mathematics for college sophomores—preliminary report*, by Professor R. V. Andree, University of Oklahoma.

The Carnegie Corporation is sponsoring an experiment in the inclusion of some mathematics other than calculus in the college sophomore course and the inclusion of examples from fields other than physics. It is suggested that about $\frac{3}{4}$ of the academic year be spent on calculus, utilizing the (polynomial) calculus contained in many freshman courses. The remaining $\frac{1}{4}$ academic year will be spent on material mainly from abstract algebra. Interested persons were encouraged to participate in the afternoon discussions or to write to the speaker expressing their opinions.

4. *What is the center of a triangle?* by Professor Arthur Bernhart, University of Oklahoma.

Each familiar notable point is introduced by a property which characterizes its central location. Besides the usual minimax properties, the points of view of physics, vector analysis, projection, perspectivity, game theory, and pursuit contribute to the exposition.

5. *Mathematics in industry*, by Mr. R. B. Rice, Phillips Petroleum Company, Bartlesville, Oklahoma.

This paper was presented at the luncheon held in conjunction with the Oklahoma Education Association. The speaker emphasized the greatly enlarged need for competent mathematicians in the petroleum industry.

The afternoon was spent in a discussion of basic problems which have arisen in the development of a new course in sophomore mathematics based on the premise that the student has had an introduction to polynomial calculus in his freshman work, and of what should and should not be included in such a course.

R. V. ANDREE, *Secretary*

ERRATA

1. A. Oppenheim, *Inequalities connected with definite Hermitian forms*, II, vol. 61, 1954, pp. 463–466. The first proof of Theorem 1 is incorrect as pointed out by the reviewer (*Math. Rev.*, vol. 16, 1955, p. 328). The second proof (by the referee) is correct.

2. L. E. Bush, *The William Lowell Putnam Mathematical Competition*, vol.

62, 1955, pp. 558–564. On page 560, problem 3, for $x_n = \sum_{i=n+1}^{\infty} x_i$ read $x_n \leqq \sum_{i=n+1}^{\infty} x_i$.

3. E. T. Parker and A. M. Mood, *Some balanced Howell rotations for duplicate bridge sessions*, vol. 62, 1955, pp. 714–716. The four-table rotation is not balanced as claimed. In the table on page 714, the cyclic permutations at Tables 2, 3, and 4 should begin with 5, 3, and 2 instead of 4, 6, and 7 respectively. The last paragraph of Section 1 should have 3, 4, 5 changed everywhere to 2, 3, 7.

4. Tomlinson Fort, *Linear difference equations and the Dirichlet series transform*, vol. 62, 1955, pp. 641–645. At the bottom of page 642, in II, instead of:

$$\sum_{t=0}^{\infty} \frac{m^{-s(t+1)}}{m^{-s}-1} \quad \text{read} \quad \sum_{t=0}^{\infty} \frac{m^{-s(t+1)}}{m^{-s}-1};$$

$$\frac{m^{-s}}{(m^s-1)^2} \quad \text{read} \quad \frac{m^s}{(m^s-1)^s}.$$

CALENDAR OF FUTURE MEETINGS

Thirty-seventh Summer Meeting, University of Washington, Seattle, Washington, August 20–21, 1956.

Fortieth Annual Meeting, University of Rochester, Rochester, New York, December 29, 1956.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Geneva College, Beaver Falls, Pennsylvania, April 28, 1956.

ILLINOIS, Eastern Illinois State College, Charleston, May 11–12, 1956.

INDIANA, Wabash College, Crawfordsville, May 5, 1956.

IOWA, Grinnell College, Grinnell, April 20–21, 1956.

KANSAS, University of Wichita, April 21, 1956.

KENTUCKY, University of Kentucky, Lexington, April 28, 1956.

LOUISIANA-MISSISSIPPI, McNeese State College, Lake Charles, Louisiana, February 17–18, 1956.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK, Stevens Institute of Technology, Hoboken, New Jersey, April 28, 1956.

MICHIGAN, University of Michigan, Ann Arbor, March 24, 1956.

MINNESOTA, Augsburg College, Minneapolis, April 28, 1956.

MISSOURI, Fontbonne College, St. Louis, April 21, 1956.

NEBRASKA, University of Nebraska, Lincoln, April 21, 1956.

NORTHEASTERN

NORTHERN CALIFORNIA

OHIO, Oberlin College, Oberlin, April 14, 1956.

OKLAHOMA, Tulsa, March 31 or April 2, 1956.

PACIFIC NORTHWEST, Oregon State College, Corvallis, June, 1957.

PHILADELPHIA

ROCKY MOUNTAIN, University of Utah, Salt Lake City, May 4–5, 1956.

SOUTHEASTERN, University of Georgia, Athens, March 16–17, 1956.

SOUTHERN CALIFORNIA, Pomona College, Claremont, March 17, 1956.

SOUTHWESTERN, New Mexico College of Agriculture and Mechanical Arts, Las Cruces, April, 1956.

TEXAS, Southwest Texas State Teachers College, San Marcos, April 20–21, 1956.

UPPER NEW YORK STATE, Alfred University, Alfred, April 28, 1956.

WISCONSIN, Marquette University, Milwaukee, May, 1956.

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512 pages 6" x 9" Published January, 1956

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320 pages 5¼" x 8¼" Published January, 1956

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200 pages 8½" x 11" Published January, 1956

MATHEMATICS OF FINANCE

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